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Micho Đurđevich
Instituto de Matemáticas, UNAM
Area de la Investigación Científica
Circuito Exterior, Ciudad Universitaria
México DF, CP 04510, México

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Micho Đurđevich
Instituto de Matemáticas, UNAM
Area de la Investigación Científica
Circuito Exterior, Ciudad Universitaria
México DF, CP 04510, México

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Micho Đurđevich
Instituto de Matemáticas, UNAM
Area de la Investigación Científica
Circuito Exterior, Ciudad Universitaria
México DF, CP 04510, México

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CATEGORICAL FRAMEWORKS FOR QUANTUM SPACES, GROUPS AND BUNDLES

Micho Đurđevich
Instituto de Matemáticas, UNAM
Area de la Investigación Científica
Circuito Exterior, Ciudad Universitaria
México DF, CP 04510, México
micho@matem.unam.mx
www.matem.unam.mx/~micho

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Abstract

We present a categorical framework for constructing and studying quantum spaces, groups and bundles, in which the concept of space is enhanced to include a variety of new purely quantum phenomena, all intuitively interpretable as manifestations of some internal oscillating collectivity-like modes of the geometry. Thanks to a higher flexibility, there are more symmetries, and in general more transformations between quantum spaces, so we can coherently formulate a concept of group, principal bundle, diagonal map and fibered product, which is not possible to achieve within the standard formulation. A number of illustrative examples and geometric and algebraic constructions are presented. Important related categorical and C*-algebraic constructions are discussed.

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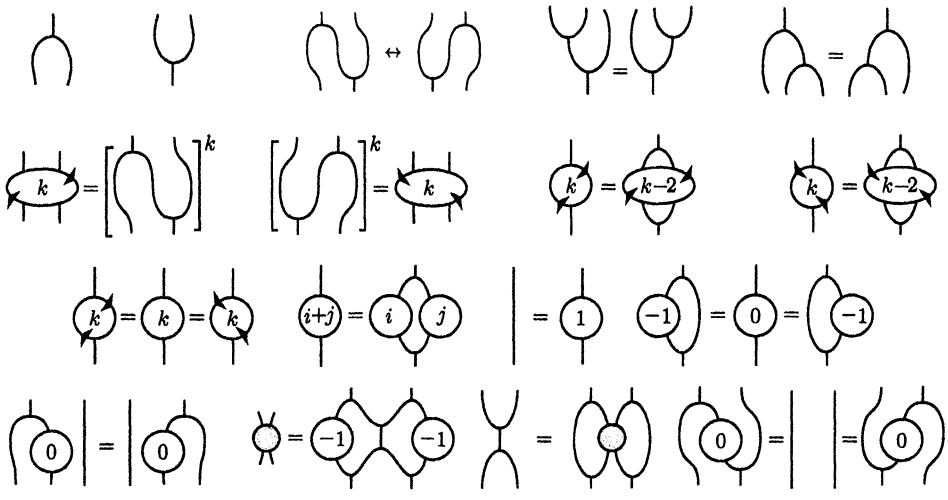
1 Introduction

In constructing a new theory, going beyond the limits of an old mathematical realm, a crucial role is played by principles of simplicity, versatility and conceptual soundness and richness, as well as those of internal coherency. A desired ideal scenario is that a new theory provides a way to see the old mathematical realm, as a particular special case, while at the same time preserving and enhancing its most interesting and profound features.

Quantum geometry, in its diverse realizations [4, 24, 27], provides an array of ideas and methods to go beyond the classical concept of space, as understood in terms of collections of points equipped with some additional structure. The additional structure being given by measurable sets, topology or smooth atlas—in the most important classical levels of geometrical insight.

In extending the idea of space, a good strategy is to operate in the spirit of Erlangen Program by Felix Klein. To try to build geometry in resonance with the concepts of symmetry, harmony, transformation. That is to say, it is natural to expect that a new world would admit objects similar as groups, homogeneous spaces and principal bundles. But apriori we do not know how the new universe would look like, and in principle there could exist many non-equivalent generalizations.



So it is good to develop a kind of 'meta-language' which would allow to express the mentioned intuitive ideas, in terms that are independent of the concrete nature of the generalization.

One such a language for groups can be built on diagrams [7, 8] and a simple alphabet consisting of two symbols representing the product and the diagonal map. In the illustration above, some basic constructs-words, expressions and formulae based on these two primordial symbols are listed.

By a natural extension of this basic language, one can then construct a diagrammatic vocabulary for actions and bundles.

A conceptual problem might arise, when it appears unclear how to express diagrams in a given quantum category. This might be an indication of an *inadequacy* of the given category to express the complete variety of geometrical phenomena. And naturally we are led to search for new expressions, perhaps somehow 'hidden' within a given conceptual framework.

These new expressions of transformation and symmetry can be realized in a systematic way, by extending and modifying the morphisms of the initial not-so-flexible category. The aim of this study is to present general methods for constructing such categorical extensions and modifications. Along the way, we shall present various examples, and discuss related conceptual ideas from quantum physics, philosophy and arts. The paper will also provide the tools for the fourth part [12] of the quantum principal bundles series [9, 10, 11].

The paper is organized in the following way. We begin by outlining the general conceptual framework for our considerations, and explain in detail why the standard formulation of quantum groups is inadequate for the development of a truly geometric theory. To establish the motivations and philosophical ground for our considerations, we shall present diverse arguments—of algebraic, geometric and intuitive nature.

We then proceed with various categorical considerations in Section 3. We shall extensively use the language of objects, morphisms, functors and natural transformations, and we refer to [14, 17] as both the reference and the inspirational source on the matter. The main idea for all constructions of this section is to extend and deepen the notion of morphism between quantum spaces, constructing a more elaborate category over the same objects. In such a transformed framework, we have more flexibility to manip-

ulate objects, thus overcoming possible inherent restrictions of the initial mathematical realm.

We first consider a very simple and 'purely algebraic' situation, where one 'blueprint' category acts on another category $\mathcal{C}$ in the appropriate way. The blueprint category is interpreted as containing the additional symmetry information, which is implemented in $\mathcal{C}$ via the mentioned action. The objects of $\mathcal{C}$ can be thought as 'quantum spaces'. A simple construction we present enables us to extend the notion of morphism in the category $\mathcal{C}$. This construction is further elaborated in a special case when the blueprint category consists of natural numbers equipped with appropriate (for example, all or some) injective maps. Intuitively, we can imagine the action of a number n to a given space X as moving it to its own appropriate 'excitation mode' where the internal geometry oscillates. So new morphisms appearing in the game are those that provide relations between the 'excited' spaces. A special attention is payed to the case when we can construct a limit object, which is a kind of universal envelope for all these 'vibrational modes'. The blueprint category can also be viewed as a simplification of the universal diagrammatic category.

In Section 4, we shall further generalize all these considerations and present a simple categorical formulation of such a universal oscillation envelope, in terms of a single covariant functor acting on the initial category $\mathcal{C}$, and a couple of natural transformations. We shall develop various constructions in considerable detail, both for the sake of completeness and to prepare for the principal examples of spaces and bundles [12].

In Section 5 we illustrate the general constructions on several noncommutative (including C*-algebraic) examples. The main example, and the inspiration for this study, is the category of quantum spaces introduced in [27]. We then consider a simple modification of this construction, in the very spirit of K-theory, to allow more freedom via combining everything with infinite matrices. This gives us a different category, in which isomorphism of C*-algebras as objects, means the strong Morita equivalence between them. This category possesses a number of remarkable properties, in particular it allows a natural introduction of diagonal map $A \otimes A \to A$ for every quantum space $X \leftrightarrow A$. This allows us to introduce the concept of a quantum group, as a categorical realization of the diagrammatic framework

developed in [7, 8]. In these 'true' quantum groups, all relevant maps (including the co-unit* and the antipode) are geometrical morphisms. Besides including all the standard quantum groups, our quantum formulation allows for objects that are not interpretable as quantum group/Hopf algebras in a standard way. Such as a quantum torus, a quantum plane (based on the Heisenberg-Weyl algebra), or any geometrically quantized symplectic manifold, for example. They do not allow a Hopf algebra structure due to the non-existence of characters in the non-commutative sector. And having such a natural concept of a group, we can consider the fibered spaces, and in particular quantum principal bundles [12]. Another very nice feature of this category is that it allows the fibered products of quantum bundles, so the entire formulation of quantum principal bundles can be completely viewed as a geometrical realization of the Galois extension concept.

We shall also present an example based on an idea of 'infinite reflections' of a quantum space in itself. This example is particularly suitable for considerations involving symmetries between a given C*-algebra A and its opposite algebra A^{op}, including the Tomita-Takesaki modular theory.

In Section 6 some concluding remarks of philosophical and conceptual nature are made. The paper ends with several Appendices, in which a technical material related to main considerations has been collected. In the first Appendix, we outline the construction and basic properties of multiplier algebras and discuss automorphisms and their limits. We then consider certain more technical issues that are used through the text. This includes the commutative Banach algebras and polar decomposition of closed operators in a Hilbert space. Finally, we present the construction of a blueprint category of numbers, in terms of certain binary sequences. This is used in the example of 'infinite reflections' category.

Our entire exposition is in the form of a 'sketchbook'. The constructions and ideas should be understood as connected, but not linearly, and possessing their own autonomy and freedom to transform—in order to adapt to specific quantum-geometrical examples and contexts. Formalism, be it axiomatic, categorical, algebraic or analytic, and intuitive geometrical presentations and visualizations, should always be considered as fundamental

*The co-unit understood as a unified entity, a natural 'inactivity' zero circular morphism in the diagrammatic formulation.

mutually complementary components of every quantum theory of geometry.

Acknowledgement: I am very indebted to Prof. Zbigniew Oziewicz for numerous discussions and constructive remarks, and for his constant encouragements and enthusiastic support during the preparation of these 3 manuscripts.

2 Geometrical Obstacles in Standard Formulation

Cuadruple Obstacle

There are 4 principal and mutually interralated reasons why the standard theory of quantum groups and bundles is too restrictive, and consequently why it could be a good idea to look for its possible extensions.

Missing Important Examples

Perhaps the paradigmic example of a quantum group-like object that is not possible to include into the standard Hopf-algebraic formulation, is that of a quantum torus. It is a natural deformation of the classical 2-torus. The C*-algebra A_z is given by two unitary generators, say U and V, and a single relation

$$VU = zUV$$

where z is a given unitary complex number. The classical ($\Leftrightarrow$ commutative) case is recovered when $z = 1$. For $z \neq 1$ the algebra is non-commutative, and its behaviour is essentially different depending on whether z is rational ($\Leftrightarrow$ a root of unity) or irrational. In the former case, the algebra would be Morita equivalent to the algebra of continuous functions over the classical torus. Its center is generated by polynomials in generators U and V, in which all powers of the generators are divisible by d, the order of z. In the latter, the algebra A_z turns out to be simple, and its center is trivial.

In both cases, however, there are no classical points, and on the other hand the object behaves almost like the classical torus in diverse important

aspects. In particular it possesses a natural group of automorphisms given by SO(2) × SO(2), acting 'transitively' on A_z. It is also interpretable as (almost) trivial principal bundle over the classical circle, the structure group of which is again the circle SO(2).

The quantum torus can not be viewed as a standard (braided or non-braided) quantum group. The reason is very simple–there can not be a counit map, since it is necessarily a homomorphism (and its kernel would be an ideal of codimension one in A_z).

Conceptually and historically, a very closed example is that of the quantum plane, the algebra in this case is given by two hermitian elements say p and q, and a single relation

$$pq - qp = \frac{\hbar}{i}$$

where $i = \sqrt{-1}$ and $\hbar$ is a positive number. Or in general, the whole class of Heisenberg-Weyl algebras. The classical group of Euclidean motions E(2) naturally acts by automorphisms.

Other similar examples of group-like objects non-includable in the standard framework are classical and quantum Clifford algebras, and Cuntz algebras.

They can all be included in a diagrammatic framework of [7, 8]. However, as such this framework still does not address the geometrical nature of these objects, providing a kind of a purely algebraic 'fix'.

Classical Points Obstacle

The standard quantum groups are very inhomogeneous objects. Unlike their classical siblings, being metaphors of a supreme realization of geometrical harmony, every standard quantum group G is always 'divided' in two parts: the classical part G_{cl} consisting of classical points (the neutral element/counit is always there), and the complement which is 'pointless'. The classical part G_{cl} is always a group, and the group structure is induced from the quantum group structure of G. So G_{cl} is actually a very important subgroup of G.

This inhomogeneity becomes explicitly visible in situations where automorphisms of G as a quantum space appear. For example, if we consider the theory of locally-trivial quantum principal G bundles over classical smooth manifolds [9], then it turns out that all these bundles are in a natural one-to-one correspondence with classical G_{cl}-bundles over the same base manifolds. At the intuitive-geometric level, it is easy to understand this phenomenon:

When we construct a locally trivial bundle, we encode its strucure in a system of transition functions from one trivialization to another, neighbouring one. These transition functions preserve the structure of vertical fibers, each one of them being represented by G. So they respect the classical-quantum division, and thus map the classical part G_{cl} into itself. This allows us to restrict to these classical parts, and construct the corresponding G_{cl} bundle. Because of the covariance of the transition functions, they are completely determined by their restrictions to the corresponding classical parts. If P is the quantum principal G-bundle associated to a given quantum cocycle, then the corresponding classical bundle P_{cl} will be interpretable as the classical part of P.

Non-geometrical Formulation

The standard formulation lacks the appropriate geometrical interpretation of the principal maps appearing in the game. For example, the quantum analogue of the inverse map, is the antipode map $\kappa\colon \mathcal{A} \to \mathcal{A}$ acting in the appropriate *-algebra $\mathcal{A}$. In the standard Hopf algebra context, this map is necessarily antimultiplicative, that is $\kappa(ab) = \kappa(b)\kappa(a)$ for every $a, b \in \mathcal{A}$ and it is also not very compatible with the *-structure, as it satisfies a twisted condition $\kappa(\kappa(a^*)^*) = a$ for every $a \in \mathcal{A}$.

Such a situation is incompatible with a geometrical interpretation of morphisms, where morphisms are played by certain *-homomorphims of non-commutative *-algebras representing the quantum spaces (and acting in the opposite directions).

Non-existence of Important Maps

A quantum counterpart for the diagonal map does not exist in the standard formulation. The diagonal map is conceptually very important in the

context of classical groups, as it provides a dual-complementary morphism to the product in the group. This opens up a way for an elegant diagramatical formulation to take place [7, 8] allowing us to construct explicitly all relevant objects in terms the group product and the diagonal map. In the quantum context we would like to use say a C*-algebra A to represent a quantum group G, and the group structure would correspond to a certain morphism $\phi\colon A \to A \otimes A$ between the corresponding C*-algebras. Then to have everything coherently represented, we would need to have the diagonal map, represented by a similar morphism $\Delta\colon A \otimes A \to A$.

However, the standard representation of such a diagonal map as *the product* is inherently flawed: It is not an algebra map, and it is even not continous, so it can not be defined at the level of C*-algebraic tensor products. We can only restrict ourselves to certain *-subalgebras $\mathcal{A}$ of A and consider purely algebraic tensor products, and then postulate that the product $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the quantum counterpart of the diagonal map. However, in the non-commutative context the product map will not be multiplicative, nor will it intertwine the *-structures on $\mathcal{A} \otimes \mathcal{A}$ and $\mathcal{A}$.

As a consequence of this, the theory of quantum principal bundles built on grounds from such a restrictive and inconsistent realm, also lacks necessary geometrical flexibility. In particular, it is not possible to introduce in a simple and geometric way the fibered product of bundles over the same quantum space. In particular, we can not make a full parallel between the geometrical theory of bundles and algebraic modes of Galois theory. This also reflects on constructions involving analogs of the gauge bundle and gauge transformations.

3 Basic Categorical Considerations

3.1 Gauge Symmetries Category

Let $\mathcal{N}$ be a small exact monoidal category with $\cdot$ as the monoidal product. We shall denote by $e_\varkappa$ the identity transformation of $\varkappa \in \mathcal{N}$. So we have

$$e_{\varkappa\cdot\lambda} = e_\varkappa \cdot e_\lambda \tag{3.1}$$

for every $\varkappa, \lambda \in \mathcal{N}$. We would like to play a kind of *permutations game*, so it will be assumed that there exists a natural involutive transformation σ between the product functor $\cdot\colon \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ and its opposite $\cdot_{\mathrm{op}}$. In other words, for every $\varkappa, \lambda \in \mathcal{N}$ there exist a morphism $\sigma_{\varkappa\lambda}\colon \varkappa \cdot \lambda \to \lambda \cdot \varkappa$ such that

$$\begin{array}{ccc} a \cdot b & \xrightarrow{f \cdot g} & c \cdot d \\ {\scriptstyle \sigma_{ab}}\downarrow & & \downarrow{\scriptstyle \sigma_{cd}} \\ b \cdot a & \xrightarrow[g \cdot f]{} & d \cdot c \end{array} \tag{3.2}$$

for every $a, b, c, d \in \mathcal{N}$ and morpshims $f\colon a \to c$ and $g\colon b \to d$. The involutivity means that

$$\sigma_{\varkappa\lambda}\sigma_{\lambda\varkappa} = \mathrm{id}_{\varkappa\cdot\lambda}. \tag{3.3}$$

In particular we see that all $\sigma_{\varkappa\lambda}$ are isomorphisms between $\varkappa \cdot \lambda$ and $\lambda \cdot \varkappa$.

We shall also assume that our twists satisfy the following pentagonal multiplicativity property

$$\begin{aligned} \sigma_{\varkappa\cdot\lambda x} &= (\sigma_{\varkappa x} \cdot e_\lambda)(e_\varkappa \cdot \sigma_{\lambda x}) \\ \sigma_{x\varkappa\cdot\lambda} &= (e_\varkappa \cdot \sigma_{x\lambda})(\sigma_{x\varkappa} \cdot e_\lambda) \end{aligned} \tag{3.4}$$

for every $\varkappa, \lambda, x \in \mathcal{N}$.

Remark 3.1. The morphisms $\sigma_{\varkappa\lambda}$ can be thought of as ways to make twists between objects. However we shall not require any braid-type property for those twists, although in a variety of interesting scenarios they will satisfy it. In the spirit of our interpretation it could look quite unnatural to not postulate the braid equation, however we do not like to miss possibly interesting examples, and in many cases the braid equation is a consequence of more fundamental additional properties. In fact, our blueprint category can be viewed as a derived object of the universal diagrammatic category of [7, 8].

Furhtermore it will be assumed that $\mathcal{N}$ possesses a *neutral object* $\varnothing$ so that

$$\varnothing \cdot \varkappa = \varkappa \cdot \varnothing = \varkappa \tag{3.5}$$

naturally holds for every $\varkappa \in \mathcal{N}$. Clearly, the neutral object is uniquely determined by the above property. By the naturality with respect to $\mathcal{N}$ we mean that

$$e_{\varnothing} \cdot \phi = \phi \cdot e_{\varnothing} = \phi \tag{3.6}$$

for every morphism ϕ in $\mathcal{N}$, as well as

$$\sigma_{\varnothing\varkappa} = \sigma_{\varkappa\varnothing} = e_{\varkappa}. \tag{3.7}$$

We could also assume that $\varnothing$ is the *first object* of $\mathcal{N}$ so that for every object μ in $\mathcal{N}$ there exists a unique morphism $\iota_{\mu}\colon \varnothing \to \mu$. In particular, we would have

$$\psi\iota_{\mu} = \iota_{\nu} \tag{3.8}$$

for every $\mu, \nu \in \mathcal{N}$ and a morphism $\psi\colon \mu \to \nu$. It is also worth noticing that, in this case

$$\iota_{\mu} \cdot \iota_{\nu} = \iota_{\mu\cdot\nu}. \tag{3.9}$$

These inclusion morphisms ι_{μ} as well as the twist morphisms $\sigma_{\varkappa\lambda}$ will be of central importance for our further considerations.

Lemma 3.1. *The following identity holds*

$$\sigma_{\varkappa\lambda}(e_{\varkappa} \cdot \iota_{\lambda}) = \iota_{\lambda} \cdot e_{\varkappa} \tag{3.10}$$

for every $\varkappa.\lambda \in \mathcal{N}$.

Proof. This follows from the functoriality of the twist, and the triviality (3.7) of its action on the products with $\varnothing$. □

We shall now introduce a subcategory of $\mathcal{N}$ which will play an important role of categorical 'gauge transformations'.

Definition 3.2. Any morphism of the form $e_{\varkappa}\cdot\iota_{\mu}\cdot e_{\lambda}\colon \varkappa\cdot\lambda \to \varkappa\cdot\mu\cdot\lambda$ is called an *elementary morphism.* Let $\mathcal{M}$ be a subcategory of $\mathcal{N}$ generated by the same objects and the elementary morphisms. By definition, every morphism of $\mathcal{M}$ is a composition of finitely many elementary morphisms. For a given morphism from $\mathcal{M}$, we shall define its *degree* as the minimal number of these elementary morphisms appearing in its all possible decompositions. The identity morphisms are considered as (the only ones) having degree zero.

It is easy to see that $\mathcal{M}$ is a monoidal subcategory of $\mathcal{N}$. We can also say that $\mathcal{M}$ is obtained by playing with inclusions ι_μ and asking what would be the minimal monoidal subcategory of $\mathcal{N}$ containing all of them. In general, the twist morphisms will not belong to $\mathcal{M}$, and so we could consider the (necessarily monoidal) subcategory generated by $\mathcal{M}$ and all possible twists. It would be natural to assume that in such a way we arrive to the full $\mathcal{N}$. This is a kind of *minimality property* for $\mathcal{N}$.

We shall principally be concerned with the scenarios when $\mathcal{M}$ admits *equalizers*. By definition, if $f\colon \mu \to \varkappa$ and $g\colon \mu \to \lambda$ are morphisms from $\mathcal{M}$, an equalizer for f, g is an object ν together with morphisms $u\colon \varkappa \to \nu$ and $v\colon \lambda \to \nu$ from $\mathcal{M}$ such that

$$\begin{array}{ccc} \mu & \xrightarrow{f} & \varkappa \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ \lambda & \xrightarrow[v]{} & \nu \end{array} \tag{3.11}$$

and we would say that ν with u and v equalizes μ with f and g.

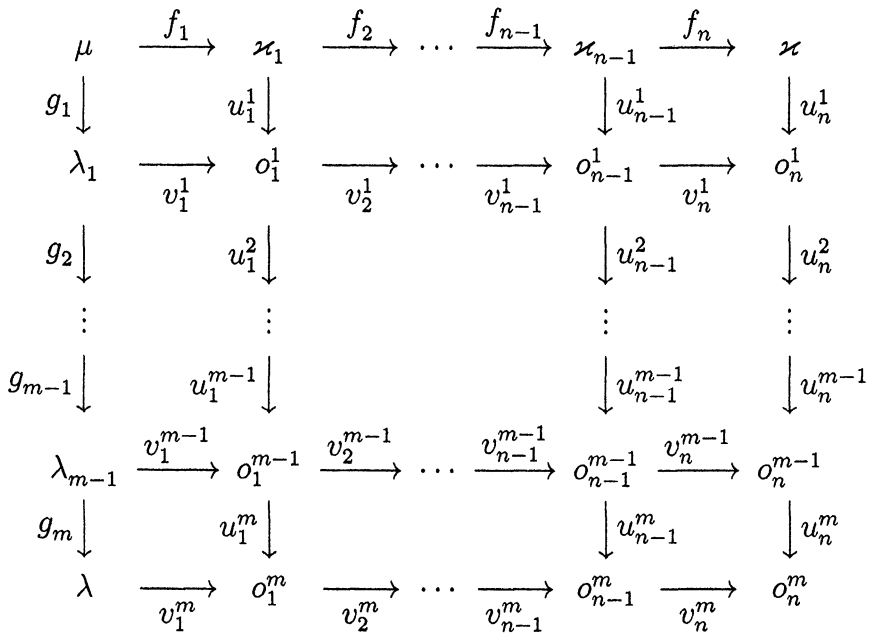
Lemma 3.2. *If elementary morphisms can be equalized with elementary morphisms, then $\mathcal{M}$ admits equalizers.*

Proof. Let us assume that

$$f = f_1 \cdots f_n\colon \mu \to \varkappa \qquad g = g_1 \cdots g_m\colon \mu \to \lambda$$

are morphisms from $\mathcal{M}$ decomposed into elementary ones. If equalizers property holds at the level of elementary morphisms, then we can construct

a commutative diagram of the form



and so $\nu = o_n^m$ with $u = u_n^1 \cdots u_n^m$ and $v = v_1^m \cdots v_n^m$ equializes μ with f and g. □

Remark 3.3. In order to further complicate things, we shall use, depending on the context, both left-to-right and right-to-left interpretation for the composition of morphisms.

Remark 3.4. The objects of the category $\mathcal{N}$ form a unital associative semigroup, relative to the monoidal multiplication. If $\varkappa$ is invertible, that is, if there exists $\varkappa^{-1}$ such that $\varkappa^{-1} \cdot \varkappa = \varkappa \cdot \varkappa^{-1} = \varnothing$, then $\varkappa$ and $\varnothing$ are isomorphic (via left/right translations by $\varkappa$). So if $\mathcal{N}$ is a group, the category trivializes as all objects are isomorphic to $\varnothing$ and there exists a unique morphism between every two objects.

3.2 Symmetry Category Actions

We shall now consider situations when the category $\mathcal{N}$ 'acts' in some natural way, on another category $\mathcal{C}$.

Definition 3.5. Let $\mathcal{C}$ be a category. A left action of $\mathcal{N}$ on $\mathcal{C}$ is a covariant functor $\circ\colon \mathcal{N} \times \mathcal{C} \to \mathcal{C}$ such that

$$\varnothing \circ x = x \tag{3.12}$$

identically on $\mathcal{C}$, and the diagram

$$\begin{array}{ccc} \mathcal{N} \times \mathcal{N} \times \mathcal{C} & \xrightarrow{\mathrm{id} \times \circ} & \mathcal{N} \times \mathcal{C} \\ {\scriptstyle \cdot \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathcal{N} \times \mathcal{C} & \xrightarrow[\circ]{} & \mathcal{C} \end{array} \tag{3.13}$$

is commutative. Similarly, a functor $\circ\colon \mathcal{C} \times \mathcal{N} \to \mathcal{C}$ is called *right action* of $\mathcal{N}$ on $\mathcal{C}$ iff the diagram

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{N} \times \mathcal{N} & \xrightarrow{\circ \times \mathrm{id}} & \mathcal{C} \times \mathcal{N} \\ {\scriptstyle \mathrm{id} \times \cdot}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathcal{C} \times \mathcal{N} & \xrightarrow[\circ]{} & \mathcal{C} \end{array} \tag{3.14}$$

is commutative and

$$x \circ \varnothing = x \tag{3.15}$$

for every object x in $\mathcal{C}$.

Remark 3.6. It is clear that $\mathcal{N}$ via the product $\cdot$ acts on itself, both on right and left.

Remark 3.7. It is also clear that the left and right actions concepts are interchangeable: A left action of $\mathcal{N}$ on $\mathcal{C}$ is the right action of $\mathcal{N}$ equipped with the opposite product $\cdot_{\mathrm{op}}$ on the same category $\mathcal{C}$, and vice versa.

Let us now consider a given category $\mathcal{C}$ and a left action $\circ$ of $\mathcal{N}$ on $\mathcal{C}$. We shall denote by i_a the identity morphism of an object a of $\mathcal{C}$. So that we have

$$e_\varkappa \circ i_a = i_{\varkappa \circ a} \tag{3.16}$$

for every $\varkappa \in \mathcal{N}$.

For $a, b \in \mathcal{C}$ and $\varkappa \in \mathcal{N}$ let $\mathrm{mor}(a,b)^{\varkappa}$ be the set of all morphisms $\psi\colon a \to \varkappa \circ b$ and define

$$\mathrm{mor}(a,b) = \bigsqcup_{\varkappa \in \mathcal{N}} \mathrm{mor}(a,b)^{\varkappa} \tag{3.17}$$

that is, we consider the disjoint union of all of them. We thus arrive at the main theme of this subsection: a natural modification of the category $\mathcal{C}$, where we can keep the objects of $\mathcal{C}$, but the morphisms between a and b are represented by elements of $\mathrm{mor}(a,b)$. We have to define the composition of arrows, and we define it to be

$$\varphi \diamond \psi = \varphi(e_{\varkappa} \circ \psi) \tag{3.18}$$

for $\varphi \in \mathrm{mor}(a,b)^{\varkappa}$ and $\psi \in \mathrm{mor}(b,c)$. This gives the following compositions

$$\mathrm{mor}(a,b)^{\varkappa} \times \mathrm{mor}(b,c)^{\lambda} \rightsquigarrow \mathrm{mor}(a,c)^{\varkappa \cdot \lambda}. \tag{3.19}$$

Lemma 3.3. *The defined composition is associative, in other words if a, b, c, d are objects of $\mathcal{C}$ then*

$$(\varphi \diamond \psi) \diamond \eta = \varphi \diamond (\psi \diamond \eta) \tag{3.20}$$

for every $\varphi \in \mathrm{mor}(a,b)$, $\psi \in \mathrm{mor}(b,c)$ and $\eta \in \mathrm{mor}(c,d)$.

Proof. If $\varphi \in \mathrm{mor}(a,b)^{\varkappa}$ and $\psi \in \mathrm{mor}(b,c)^{\lambda}$ then for every $\eta \in \mathrm{mor}(c,d)$ we have

$$(\varphi \diamond \psi) \diamond \eta = \varphi(e_{\varkappa} \circ \psi) \diamond \eta = \varphi(e_{\varkappa} \circ \psi)(e_{\varkappa \cdot \lambda} \circ \eta) = \varphi(e_{\varkappa} \circ \psi)\big[(e_{\varkappa} \cdot e_{\lambda}) \circ \eta\big]$$
$$\varphi(e_{\varkappa} \circ \psi)\big[e_{\varkappa} \circ (e_{\lambda} \circ \eta)\big] = \varphi\big[e_{\varkappa} \circ \big(\psi(e_{\lambda} \circ \eta)\big)\big] = \varphi \diamond (\psi \diamond \eta)$$

that is, our product is associative. □

As a matter of straightforward verification

$$i_a \diamond \varphi = \varphi \diamond i_b = \varphi. \tag{3.21}$$

Therefore we can conclude that objects of $\mathcal{C}$, equipped with morphisms $\mathrm{mor}(a,b)^{\varkappa}$ and composition $\diamond$ form a new category $\mathcal{C}_{\infty}$. Every morphism

from $\mathcal{C}_\infty$ caries a 'charge' and the composition of morphisms multiplies the charges, in accordance with (3.19). The initial category $\mathcal{C}$ is realized as a subcategory of $\mathcal{C}_\infty$, corresponding to the restriction to morphisms of the neutral charge $\varnothing$.

We shall now extend the action of $\mathcal{N}$ from $\mathcal{C}$ to $\mathcal{C}_\infty$. To do this, we only have to define the extension of the action of morphisms, since the objects of both categories are the same. So if $f\colon \varkappa \to \lambda$ and $\psi \in \mathrm{mor}(a,b)^\varrho$ we define $f \circ \psi \in \mathrm{mor}(\varkappa \circ a, \lambda \circ b)^\varrho$ by the following expression

$$f \circ \psi = (\sigma_{\lambda\varrho} \circ i_b)(f \circ \psi)\colon \varkappa \circ a \to \varrho \circ (\lambda \circ b) \tag{3.22}$$

where on the right-hand side $f \circ \psi$ is interpreted in the standard sense, as a morphism in $\mathcal{C}$.

Proposition 3.4. *The defined action is indeed, an action of $\mathcal{N}$ on $\mathcal{C}_\infty$, extending the action of $\mathcal{N}$ on $\mathcal{C}$.*

Proof. Let us see how the new action relates to the composition of morphisms. If f and ψ are as above, and $\varphi \in \mathrm{mor}(b,c)^\tau$ and $g\colon \lambda \to \mu$ then

$$\begin{aligned}(g \circ \varphi) \diamond (f \circ \psi) &= \big\{(\sigma_{\mu\tau} \circ i_c)(g \circ \varphi)\big\} \diamond \big\{(\sigma_{\lambda\varrho} \circ i_b)(f \circ \psi)\big\} = \\ \big((e_\varrho \cdot \sigma_{\mu\tau}) \circ i_c\big)\big((e_\varrho \cdot g) \circ \varphi\big)(\sigma_{\lambda\varrho} \circ i_b)(f \circ \psi) &= \big[(\sigma_{\mu\varrho} \cdot e_\tau)(e_\rho \cdot \sigma_{\mu\tau}) \circ i_c\big]\big(gf \circ (\varphi \diamond \psi)\big) \\ &= \big(\sigma_{\mu|\varrho\cdot\tau} \circ i_c\big)\big(gf \circ (\varphi \diamond \psi)\big) = (gf) \circ (\varphi \diamond \psi)\end{aligned}$$

where we have used the same symbol $\circ$ to denote the new and old actions, depending on the context. Let us check the associativity of the new action. For f and ψ as above, and $g\colon \mu \to \nu$ we have

$$\begin{aligned}\big(g \circ (f \circ \psi)\big) = g \circ \big\{(\sigma_{\lambda\varrho} \circ i_b)(f \circ \psi)\big\} &= \big(\sigma_{\nu\varrho} \circ i_{\lambda \circ b}\big)\big(g \circ (\sigma_{\lambda\varrho} \circ i_b)\big)\big(e_\mu \circ (f \circ \psi)\big) \\ &= \big(\sigma_{\nu\cdot\lambda|\varrho} \circ i_b\big)\big((g \cdot f) \circ \psi\big) = (g \cdot f) \circ \psi\end{aligned}$$

where we have again switched between two interpretations of $\circ$. It is worth noticing that in both derivations we have used the left and right compatibility between the twist and the monoidal structure of $\mathcal{N}$. $\square$

In spite of its nice properties, the category $\mathcal{C}_\infty$ turns out to be 'too simple' to provide the necessary flexibility required by intuitive geometrical considerations with quantum spaces.

We are now going to introduce an equivalence relation between the morphisms of $\mathcal{C}_\infty$ which will lead us to a much more elaborate factor-category. Let us observe that the association $\varkappa \rightsquigarrow \mathrm{mor}(a,b)^\varkappa$ is functorial, that is, it establishes a covariant functor from $\mathcal{N}$ to the category of Sets. Indeed for a morpshism $f\colon \varkappa \to \lambda$ we can associate a set map $f_*\colon \mathrm{mor}(a,b)^\varkappa \to \mathrm{mor}(a,b)^\lambda$ given by

$$f_*(\psi) = (f \circ i_b)\psi. \tag{3.23}$$

The transformations f_* will be interpreted as a kind of 'gauge symmetry' in our context. So we would like to attach the same geometrical meaning to 'gauge equivalent' morphisms. All of this naturally prompts for our next definition.

Definition 3.8. Within the introduced context, and given $a, b \in \mathcal{C}$ we shall say that morphisms ψ and φ are *elementary equivalent* if one can be obtained from another, by applying an elementary morphism. In other words $\psi = h_*(\varphi)$ or $h_*(\psi) = \varphi$ for some elementary morphism h of $\mathcal{N}$. We shall use the symbol $\sim_*$ for this relation.

The elementary-equivalence naturally generates a full equivalence relation between morphisms. Let $\sim$ be the minimal equivalence relation extending $\sim_*$. In other words $\varphi \sim \psi$ iff there exist morphisms $\zeta_1, \ldots, \zeta_n$ so that

$$\varphi \sim_* \zeta_1 \quad \zeta_1 \sim_* \zeta_2 \quad \cdots \quad \zeta_{n-1} \sim_* \zeta_n \quad \zeta_n \sim_* \psi.$$

We also include the case $n = 0$, as simply $\varphi \sim_* \psi$. There is for us a particularly interesting context, when the equivalence relation simplifies.

Lemma 3.5. *If $\mathcal{M}$ possesses equalizers, then $\psi \in \mathrm{mor}(a,b)^\varkappa$ and $\varphi \in \mathrm{mor}(a,b)^\lambda$ are equivalent iff there exists $\nu \in \mathcal{N}$ and morphisms $f\colon \varkappa \to \nu$ and $g\colon \lambda \to \nu$ from $\mathcal{M}$ such that*

$$f_*(\psi) = g_*(\varphi). \tag{3.24}$$

Proof. If the above equality holds, then by decomposing f and g into elementary morphisms, and remembering the definition of $\sim$, we see that φ and ψ are equivalent. Let us call $\backsim$ a relation between morphisms introduced as in the above lemma: $\psi \backsim \varphi$ iff there exists $\nu \in \mathcal{N}$ and morphisms $f\colon \varkappa \to \nu$ and $g\colon \lambda \to \nu$ such that (3.24) holds. If we prove that $\backsim$ is an equivalence relation, then it follows that $\sim = \backsim$. It is clear that $\backsim$ is symmetric and reflexive.

Let us assume that $\psi \backsim \varphi$ and $\varphi \backsim \zeta \in \mathrm{mor}(a,b)^{\varrho}$. Then we have ν, f, g as above, and also $\mu \in \mathcal{N}$ and $q\colon \lambda \to \mu$ and $h\colon \varrho \to \mu$ such that $q_*(\varphi) = h_*(\zeta)$. Let ϵ with $r\colon \nu \to \epsilon$ and $s\colon \mu \to \epsilon$ be an equalizer object for $g, q\colon \lambda \to \nu, \mu$. In other words $rg = sq$. We then have

$$(rf)_*(\psi) = (rg)_*(\varphi) = (sq)_*(\varphi) = (sh)_*(\zeta)$$

in other words $\psi \backsim \zeta$ and so $\backsim$ is transitive, and hence the equivalence relation coinciding with $\sim$. □

Remark 3.9. In resonance with the analogy of 'gauge transformations' we would like to construct a new category, by factorizing over this equivalence relation. To put it differently, we would like to be able to imagine all morphisms $\mathrm{mor}(a,a)^{\varkappa} \ni i_{\varkappa|a}\colon a \to \varkappa \circ a$ as representations of the identity morphism of a. The introduced equivalence relation is a natural expression of this idea.

Lemma 3.6. *Let us consider $\iota\colon \varnothing \to \lambda$ and $g \in \mathrm{mor}(b,c)^{\varkappa}$. Then the morphisms $\iota \circ g\colon b \to (\lambda \cdot \varkappa) \circ c$ and $((e_{\varkappa} \cdot \iota) \circ i_c)g\colon b \to (\varkappa \cdot \lambda) \circ c$ are elementary equivalent via the twist between $\varkappa \cdot \lambda$ and $\lambda \cdot \varkappa$.*

Proof. This is a simple application of Lemma 3.1. We have

$$\begin{aligned}(\sigma_{\lambda\varkappa} \circ i_c)(\iota \circ g) = (\sigma_{\lambda\varkappa} \circ i_c)(\iota \circ i_{\varkappa \circ c})g = (\sigma_{\lambda\varkappa} \circ i_c)\big(\iota \circ (e_{\varkappa} \circ i_c)\big)g = \\ = (\sigma_{\lambda\varkappa} \circ i_c)\big((\iota \cdot e_{\varkappa}) \circ i_c\big)g = \big((e_{\varkappa} \cdot \iota) \circ i_c\big)g.\end{aligned}$$

In other words, the mentioned morphisms are elementary equivalent. □

Lemma 3.7. *If $\varphi, \psi \in \mathrm{mor}(a,b)$ and $\varphi \sim_* \psi$ then for every $\zeta \in \mathrm{mor}(b,c)$ we have $\varphi \diamond \zeta \sim \psi \diamond \zeta$. Similarly, if $\zeta \in \mathrm{mor}(c,a)$ then $\zeta \diamond \varphi \sim_* \zeta \diamond \psi$.*

Proof. Let us assume that $\psi \in \mathrm{mor}(a,b)^{\varkappa}$ and $\varphi \in \mathrm{mor}(a,b)^{\lambda}$ with $\varphi = (h \circ i_b)\psi$ and $h\colon \varkappa \to \lambda$. Then for $\zeta\colon b \to c$ of grade ϱ we have

$$\zeta \diamond \varphi = (e_\lambda \circ \zeta)\varphi = (e_\lambda \circ \zeta)(h \circ i_b)\psi = (h \circ \zeta)\psi = \big((h \cdot e_\varrho) \circ i_b\big)\psi = \\ = (h \cdot e_\varrho)_*[\zeta \diamond \psi]$$

which shows that $\zeta \diamond \varphi$ and $\zeta \diamond \psi$ are elementary equivalent via $h \cdot e_\varrho$. Similarly, if now $\zeta \in \mathrm{mor}(c,a)$ is of grade ϱ, we have

$$\varphi \diamond \zeta = (e_\varrho \circ \varphi)\zeta = \big(e_\varrho \circ (h \circ i_b)\psi\big)\zeta = \big((e_\varrho \cdot h) \circ i_b\big)(e_\varrho \circ \psi)\zeta = \big((e_\varrho \cdot h) \circ i_b\big)(\psi \diamond \zeta) \\ = (e_\varrho \cdot h)_*[\psi \diamond \zeta]$$

in other words $\varphi \diamond \zeta$ and $\psi \diamond \zeta$ are elementary equivalent via $e_\varrho \cdot h$. □

As a consequence of the above Lemma, we conclude that

$$\varphi \sim \psi \quad \Rightarrow \quad \begin{cases} \zeta \diamond \varphi \sim \zeta \diamond \psi \\ \varphi \diamond \zeta \sim \psi \diamond \zeta \end{cases} \tag{3.25}$$

for any morphism ζ with the appropriate co/domain. Therefore, we can consistently define the composition of equivalence classes, via composing their representatives. In such a way we arrive to a new category $\mathcal{C}^*$. The objects of this category are the same as the objects of $\mathcal{C}$ and $\mathcal{C}_\infty$. The morphisms in $\mathcal{C}^*$ are equivalence classes of morphisms from $\mathcal{C}_\infty$ with the projected composition law

$$[\varphi] \diamond [\psi] = [\varphi \diamond \psi]. \tag{3.26}$$

By construction, we have a natural covariant functor from $\mathcal{C}_\infty$ to $\mathcal{C}^*$. And since $\mathcal{C}$ is a subcategory of $\mathcal{C}_\infty$, we have a covariant functor $\mathcal{C} \rightsquigarrow \mathcal{C}^*$. In general, this functor will 'forget' something about original morphisms from $\mathcal{C}$, as it factorizes through the relation $\sim$. It is interesting to ask when the morpshisms of $\mathcal{C}$ will be fully preserved, so that we can view $\mathcal{C}$ as a subcategory of $\mathcal{C}^*$.

Lemma 3.8. *If $\mathcal{M}$ possesses equalizers, then two morphisms $\varphi, \psi\colon a \to b$ in $\mathcal{C}$ are equivalent iff there exist $\mu \in \mathcal{N}$ such that*

$$\iota_\mu \circ \varphi = \iota_\mu \circ \psi. \tag{3.27}$$

In particular $\mathcal{C} \rightsquigarrow \mathcal{C}^*$ will be embedding, iff the action of $\mathcal{N}$ on $\mathcal{C}$ is such that the above equation always implies $\varphi = \psi$.

Proof. This is a straightforward consequence of Lemma 3.5 applied to the equivalence of morphisms $\varphi, \psi\colon a \to b$. □

Definition 3.10. We shall call the action of $\mathcal{N}$ on $\mathcal{C}$ injective, iff $\mathcal{C} \rightsquigarrow \mathcal{C}^*$ is embedding. So in the case of injective actions, we have an elegant way to *extend* the concept of morphism for $\mathcal{C}$. In a dual way, we shall call the action of $\mathcal{N}$ on $\mathcal{C}$ surjective, iff $\mathcal{C} \rightsquigarrow \mathcal{C}^*$ is projection, in other words, all morphisms of $\mathcal{C}^*$ are representable as morphisms from $\mathcal{C}$.

There exist an interesting collection of natural morphisms in $\mathcal{C}^*$ which, in a sense, generate the whole category extension. For every $\varkappa \in \mathcal{N}$ and $a \in \mathcal{C}$ the identity morphism $i_{\varkappa \circ a}$ can be interpreted as an element of $\mathrm{mor}(\varkappa \circ a, a)^\varkappa$, and projecting down to $\mathcal{C}^*$ we obtain a natural morphism $d_{\varkappa|a}\colon \varkappa \circ a \to a$.

Remark 3.11. For the sake of notational simplicity, we shall also use the same symbol $d_{\varkappa|a}$ for the unprojected morphism in $\mathcal{C}_\infty$.

Definition 3.12. The morphism $d_{\varkappa|a}$ is called the diagonal morphism of order $\varkappa$ for the object a.

The reason we call them diagonal morphisms will become completely transparent when we introduce various categories of quantum spaces, and consider their natural extensions. These diagonal morphisms possess a number of simple and interesting properties.

Proposition 3.9. *The diagram*

$$\begin{array}{ccc} \varkappa \circ a & \xrightarrow{d_{\varkappa|a}} & a \\ {\scriptstyle f \circ \psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \lambda \circ b & \xrightarrow[d_{\lambda|b}]{} & b \end{array} \tag{3.28}$$

is commutative. Moreover we have

$$d_{\lambda|a} d_{\varkappa|\lambda\circ a} = d_{\varkappa\cdot\lambda|a} \tag{3.29}$$

$$d_{\varnothing|a} = i_a. \tag{3.30}$$

In particular,

$$d_{\varkappa|a}(\iota_\varkappa \circ i_a) = i_a \tag{3.31}$$

for every object a in $\mathcal{C}$.

Proof. This is a matter of a straightforward verification. □

Proposition 3.10. *The category $\mathcal{C}^*$ is generated by $\mathcal{C}$ and diagonal morphisms. Specifically, every morphism $\phi\colon a \to b$ in $\mathcal{C}^*$ can be written in the form of a composition*

$$\phi = d_{\varkappa|b} \diamond \psi \tag{3.32}$$

where $\psi\colon a \to \varkappa \circ b$ is a morphism from $\mathcal{C}$.

Proof. Every morphism from $\mathrm{mor}(a,b)^\varkappa$ can be naturally viewed as a composition in $\mathcal{C}_\infty$ of a morphism in $\mathcal{C}$ between a and $\varkappa \circ b$ and the diagonal map $d_{\varkappa|b}\colon \varkappa \circ b \to b$. □

Remark 3.13. The above property (3.31) tells us that the diagonal maps are left inverses for canonical embeddings of a in $\varkappa \circ a$, as they should be.

As we shall now see, the introduced relation is compatible with the (extended) action of $\mathcal{N}$ on $\mathcal{C}_\infty$. This allows us to naturally define the action of $\mathcal{N}$ on $\mathcal{C}^*$.

Lemma 3.11. *If f is a morphism in $\mathcal{N}$ and $\psi, \varphi \in \mathrm{mor}(a,b)$ are elementary equivalent morphisms in $\mathcal{C}_\infty$ then $f \circ \psi$ and $f \circ \varphi$ are elementary equivalent, too.*

Proof. Let us consider $\psi \in \mathrm{mor}(a,b)^\varkappa$ and $\varphi \in \mathrm{mor}(a,b)^\lambda$ such that $\varphi = (h \circ i_b)\psi$ for some $h\colon \varkappa \to \lambda$. And consider $f\colon \mu \to \nu$. We then have

$$\begin{aligned} f\circ\varphi = (\sigma_{\nu\lambda}\circ i_b)(f\circ\varphi) = (\sigma_{\nu\lambda}\circ i_b)\big((f\cdot h)\circ i_b\big)(e_\mu\circ\psi) = (h\circ i_{\nu\circ b})(\sigma_{\nu\varkappa}\circ i_b)(f\circ\psi) \\ = (h \circ i_{\nu\circ b})(f \circ \psi) \end{aligned}$$

employing, as before, the same flexibility of writing for $\circ$. This shows that $f \circ \varphi$ and $f \circ \psi$ are elementary equivalent via the same morphism h. □

As a direct consequence of the above, we conclude that

$$\varphi \sim \psi \quad \Rightarrow \quad f \circ \varphi \sim f \circ \psi \tag{3.33}$$

and therefore the initial action of $\mathcal{N}$ on $\mathcal{C}_\infty$ projects naturally to the action $\mathcal{N} \times \mathcal{C}^* \rightsquigarrow \mathcal{C}^*$. We shall use the same symbol $\circ$ for this projected action. We can apply the same extension-projection procedure to the category $\mathcal{C}^*$, obtaining a new category $\mathcal{C}^{**}$. However, the second iteration of the procedure gives really nothing new.

Lemma 3.12. *The above statement is true.*

Proof. Let us consider morphisms $\tilde{f}$ and $\tilde{g}$ in the extended category $\mathcal{C}^*_\infty$ between objects a and b. Here $\tilde{f}\colon a \to \varkappa \circ b$ and $\tilde{g}\colon a \to \lambda \circ b$ are morphisms from $\mathcal{C}^*$. As such, they are representable by some $f\colon a \to \mu \circ (\varkappa \circ b)$ and $g\colon a \to \nu \circ (\lambda \circ b)$, which are morphisms from the initial category $\mathcal{C}$. Now if $\tilde{f} \sim_* \tilde{g}$ in $\mathcal{C}^*_\infty$ such that

$$(\iota \circ i_b) \diamond \tilde{f} = \tilde{g} \qquad \iota\colon \varkappa \to \lambda$$

then at the level of representatives f and g this means that

$$((e_\mu \cdot \iota) \circ i_b) f\colon a \to \mu \circ (\lambda \circ b) \quad \& \quad g$$

are equivalent as morphisms from $\mathcal{C}_\infty$. But this implies that $f \sim g$. So, equivalent morphisms in $\mathcal{C}^*_\infty$ always possess equivalent representatives in $\mathcal{C}_\infty$. This means that the natural functor $\mathcal{C}^* \rightsquigarrow \mathcal{C}^{**}$ realizes the equivalence. □

3.3 Completeness & Continuity

Now we shall focus on a very particular, but for us fundamentally important special case, when the category $\mathcal{N}$ is modeled over *natural numbers*: So the objects are natural numbers, and the morphisms are all injective maps between them. Here natural numbers are interpreted as sets

$$\varnothing \leftrightarrow 0 \quad \{0\} \leftrightarrow 1 \quad \{0,1\} \leftrightarrow 2 \quad \cdots \quad \{0,1,\ldots,n\} \leftrightarrow n+1.$$

The monoidal structure is given by the standard addition, and twist operators assume the form of the standard transpositions, acting on numbers

as sets. There is another equivalent realization of this category in terms of binary sequences whose order is a power of 2, and this is presented in detail in Appendix D. We shall denote by $[n]$ the object corresponding to a natural number n and in particular

$$[0] = \varnothing \qquad [n] \cdot [m] = [n+m].$$

We shall freely exchange between the numbers n and their representations as objects of $\mathcal{N}$. We shall also use the symbol $\bullet$ for $[1]$.

For every two natural numbers m and n with $m > n$ we shall denote by $\iota_{m|n} \colon [n] \hookrightarrow [m]$ the inclusion map. Such morphisms constitute a canonical sequence in $\mathcal{N}$ of inclusions

$$\varnothing \hookrightarrow \bullet \hookrightarrow [2] \hookrightarrow [3] \hookrightarrow [4] \hookrightarrow \cdots$$

Let us assume that $\mathcal{N}$ acts on $\mathcal{C}$. Then to every object $a \in \mathcal{C}$ we can naturally associate an infinite sequence

$$a_0 = a \hookrightarrow a_1 = [1] \circ a \hookrightarrow \cdots \hookrightarrow a_n = [n] \circ a \hookrightarrow a_{n+1} \hookrightarrow a_{n+2} \cdots \qquad (3.34)$$

of objects in $\mathcal{C}$, simply by acting on a by the inclusion sequence for numbers. We shall call the above sequence *the canonical sequence* associated to a. It is natural to ask whether the canonical sequences possess 'well-behaved' limits in our initial 'playground' category $\mathcal{C}$.

Definition 3.14. The action of $\mathcal{N}$ on $\mathcal{C}$ is called *complete*, iff there exists a covariant functor $q \colon \mathcal{C} \to \mathcal{C}$ such that for every object a in $\mathcal{C}$ its canonical sequence possesses a limit, and

$$q(a) = \varinjlim_{n \geq 0} a_n. \qquad (3.35)$$

The object $q(a)$ can be thought of as a 'quantum extension' of a. A complete action is called *continuous* iff it commutes (up to isomorphisms) with the limits of canonical sequences.

So if the action is complete, we can functorially associate to every object a its limit object $q(a)$. For a given morphism $f \colon a \to b$ in $\mathcal{C}$ we shall denote

by $q(f)\colon q(a) \to q(b)$ the corresponding morphism of extension objects. By construction, there exist morphisms $j_{n|a}\colon [n] \circ a = a_n \to q(a)$ so that

$$j_{m|a}\big((e_n \cdot \iota_{m-n}) \circ i_a\big) = j_{n|a} \qquad \forall m > n. \tag{3.36}$$

And if the action is complete and continuous, then the objects $\varkappa \circ q(a)$ and $q(a)$ are always isomorphic. Indeed, for every $\varkappa \in \mathcal{N}$ the sequence $(\varkappa \circ a_n)_{n\geq 0}$ with its morphisms multiplied by $e_\varkappa$ is just a shift of the canonical sequence $(a_n)_{n\in\mathbb{N}}$. Therefore it has the same limit objects. On the other hand, the commuting with limits property implies that the limit is isomorphic to $\varkappa \circ q(a)$. The isomorphism is uniquely determined by requiring that diagrams

$$\begin{array}{ccc} q(a) & \xrightarrow{q_{\varkappa|a}} & \varkappa \circ q(a) \\ {\scriptstyle j_{\varkappa+n|a}}\uparrow & & \uparrow{\scriptstyle e_\varkappa \circ j_{n|a}} \\ a_{\varkappa+n} & = & \varkappa \circ a_n \end{array} \tag{3.37}$$

be commutative.

Remark 3.15. When appropriate we shall interchangeably use the symbols $\cdot$ and $+$ for the monoidal product in $\mathcal{N}$.

Lemma 3.13. *We have*

$$q_{\varkappa\cdot\lambda|a} = (e_\varkappa \circ q_{\lambda|a})q_{\varkappa|a} \tag{3.38}$$

for each $\varkappa, \lambda$ in $\mathcal{N}$ and $a \in \mathcal{C}$. Moreover, if $f\colon a \to b$ is a morphism from $\mathcal{C}$ then the diagram

$$\begin{array}{ccc} q(a) & \xrightarrow{q_{\varkappa|a}} & \varkappa \circ q(a) \\ {\scriptstyle q(f)}\downarrow & & \downarrow{\scriptstyle e_\varkappa \circ q(f)} \\ q(a) & \xrightarrow{q_{\varkappa|a}} & \varkappa \circ q(a) \end{array} \tag{3.39}$$

is commutative, in other words for a given $\varkappa$, isomorphisms $q_{\varkappa|c}$ provide a natural transformation between functors $\varkappa \circ q$ and q.

Proof. The above identity follows directly by combining two copies of the diagram (3.37), one for $\varkappa$ and another for λ. The naturality of the isomorphisms follows in a similar way, as a consequence of naturality of morphisms $j_{n|a}\colon [n] \circ a = a_n \to q(a)$. □

In fact *any* sequence of maps

$$\varnothing \xrightarrow{\iota_\bullet} \bullet \xrightarrow{\zeta_2} [2] \xrightarrow{\zeta_3} [3] \xrightarrow{\zeta_4} \cdots \tag{3.40}$$

can be made equivalent in $\mathcal{N}$ to the canonical inclusion sequence $\vec{\Gamma}_k$. There exists and unique vertical arrows (permutations) so that the diagram

$$\begin{array}{ccccccccc}
\varnothing & \xrightarrow{\iota_\bullet} & \bullet & \xrightarrow{\zeta_2} & [2] & \xrightarrow{\zeta_3} & [3] & \xrightarrow{\zeta_4} & \cdots \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle p_2}\downarrow & & {\scriptstyle p_3}\downarrow & & \\
\varnothing & \xrightarrow[\iota_\bullet]{} & \bullet & \xrightarrow[\vec{\Gamma}_2]{} & [2] & \xrightarrow[\vec{\Gamma}_3]{} & [3] & \xrightarrow[\vec{\Gamma}_4]{} & \cdots
\end{array} \tag{3.41}$$

commutes.

Every such a sequence naturally induces via the action, sequences in $\mathcal{C}$ of the form

$$a \to \bullet \circ a \to [2] \circ a \to [3] \circ a \to \cdots \tag{3.42}$$

which consequently can be made equivalent to the canonical sequence for a. This implies that all the above sequences converge in $\mathcal{C}$ and have the same limit object $q(a)$. The corresponding insertion maps for (3.42) are given by

$$j_{n|a}(p_n \circ i_a)\colon [n] \circ a \longrightarrow q(a). \tag{3.43}$$

In the special case when p_n are complete inversion permutation (in other words, considering the 'mirror' of the canonical inclusion sequence) we shall use the symbol $k_{n|a}$ for the corresponding insertion maps.

Besides these one-dimensional sequences, we can associate an interesting 2-dimensional infinite array to every object $c \in \mathcal{C}$, by combining the

canonical sequence with itself:

$$\begin{array}{ccccccc}
c & \longrightarrow & \bullet \circ c & \longrightarrow & [2] \circ c & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\bullet \circ c & \longrightarrow & [2] \circ c & \longrightarrow & [3] \circ c & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
[2] \circ c & \longrightarrow & [3] \circ c & \longrightarrow & [4] \circ c & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \ddots
\end{array} \tag{3.44}$$

Here the horizontal arrows are $[m][n] \circ c \to [m][n+1] \circ c$, and vertical arrows are $[m][n] \circ c \to [m+1][n] \circ c$, both the morphisms induced by the canonical inclusion sequence in $\mathcal{N}$.

Lemma 3.14. *The square $q^2(c) = q[q(c)]$ is the limit of the above 2-dimensional diagram.*

Proof. If we consider the limits of the horizontal rows, we obtain

$$q(c) \longrightarrow \bullet \circ q(c) \longrightarrow [2] \circ q(c) \longrightarrow [3] \circ q(c) \longrightarrow \cdots$$

which is nothing but the canonical sequence for $q(c)$. By construction, its limit is $q[q(c)]$. On the other hand such a limit coincides with the limit of the original diagram. □

On the other hand, if we consider vertical columns and their limits, we obtain a sequence

$$q[c] \longrightarrow q[\bullet \circ c] \longrightarrow q\big[[2] \circ c\big] \longrightarrow q\big[[3] \circ c\big] \longrightarrow \cdots$$

which is actually the canonical sequence for c transformed by the functor q. This follows from the diagram

$$\begin{array}{ccc}
[m+n] \circ c & \xrightarrow{e_m \circ j_{n|c}} & [m] \circ q(c) \\
{\scriptstyle j_{m|n \circ c}}\downarrow & & \downarrow{\scriptstyle j_{m|q(c)}} \\
q\big[[n] \circ c\big] & \xrightarrow[q(j_{n|c})]{} & q^2(c)
\end{array} \tag{3.45}$$

We shall denote by

$$j_{mn|c}\colon c_{mn} = [m+n] \circ c \to q^2(c) \tag{3.46}$$

the corresponding insertion maps.

Proposition 3.15. *There exists a unique morphism $\sigma_c\colon q^2(c) \to q^2(c)$ making all the diagrams*

$$\begin{array}{ccc} [m+n] \circ c & \xrightarrow{j_{mn|c}} & q^2(c) \\ {\scriptstyle \sigma_{mn} \circ i_c}\downarrow & & \downarrow{\scriptstyle \sigma_c} \\ [n+m] \circ c & \xrightarrow[j_{nm|c}]{} & q^2(c) \end{array} \tag{3.47}$$

commutative. These morphisms are isomorphisms and constitute a natural automorphism of the functor q^2.

Proof. Transpositions $[m+n] \circ c \to [n+m] \circ c$ realize a natural symmetry of the 2-dimensional array defining $h^2(c)$. □

Lemma 3.16. *The limit of the diagonal sequence*

$$a \to a_2 = [2] \circ a \to a_4 = [4] \circ a \to \cdots \tag{3.48}$$

is naturally isomorphic to $q(a)$, via the insertion maps $\partial_{n|a}\colon a_{2n} \to q(a)$ given by

$$\partial_{n|a} = j_{2n|a}\{r_n \circ i_a\} \tag{3.49}$$

where r_n are shuffling permutations acting on $[2n]$ defined by

$$r_n\colon (1, \ldots, n, n+1, \ldots, 2n) \mapsto (1, n+1, \ldots, n, 2n)$$

in other words we put together matching elements of the first and the second linear group of n numbers.

Proof. We have the following commutative diagram

$$\begin{array}{ccc} [2n] & \xrightarrow{\iota_{2n|2n+2}} & [2n+2] \\ {\scriptstyle r_n}\uparrow & & \uparrow{\scriptstyle r_{n+1}} \\ [2n] & \xrightarrow[\iota_n \cdot \iota_n]{} & [2n+2] \end{array} \tag{3.50}$$

which implies that the sequence of isomorphisms $r_n \circ i_a$ intertwines the diagonal sequence (3.48) and the (even part of the) canonical sequence for a. Hence, the diagonal sequence has $q(a)$ as its limit, with the insertion morphisms defined as above. □

Remark 3.16. The morphisms from the diagonal sequence to objects of $\mathcal{C}$ are uniquely extendible to morphisms on the entire 2-dimensional diagram.

Proposition 3.17. *There exist natural isomophisms $\mu_c\colon q^2(c) \to q(c)$ realizing a natural transformation between functors q^2 and q, and are uniquely defined by commutative diagrams*

$$\begin{array}{ccc} c_{2n} = [n] \circ c_n & \xrightarrow{e_n \circ j_{n|c}} & [n] \circ q(c) \\ \partial_{n|c} \downarrow & & \downarrow k_{n|q(c)} \\ q(c) & \xleftarrow[\mu_c]{} & q[q(c)] \end{array} \tag{3.51}$$

that is, the intertwining of the corresponding insertion maps.

Proof. This is a consequence of the above results, on the limits of the diagonal as well as 2-dimensional sequence associated to c. Since the diagonal sequence converges to both $q(c)$ and $q^2(c)$ they must be isomorphic, and the isomorphism is uniquely defined by (3.51), expressing the universal property of limits. □

We shall now have a look at the behaviour of the functor q when applied to product objects $[n] \circ a = a_n$. If we act on the canonical sequence for a by $[n]$, we obtain a sequence converging to $[n] \circ q(a)$. By applying the twist operators σ_{nk} on every element $[n+k] \circ a$ of this new sequence, we realize the equivalence with the canonical sequence for $[n] \circ a$. Therefore, there exists a natural isomorphism $l_{n|a}\colon [n] \circ q(a) \to q([n] \circ a)$ uniquely determined by

$$\begin{array}{ccc} [n+k] \circ a & \xrightarrow{e_n \circ j_{k|a}} & [n] \circ q(a) \\ \sigma_{nk} \circ i_a \downarrow & & \downarrow l_{n|a} \\ [k+n] \circ a & \xrightarrow[j_{k|[n]\circ a}]{} & q([n] \circ a) \end{array} \tag{3.52}$$

These morphisms establish an equivalence between functors $a \rightsquigarrow [n] \circ q(a)$ and $a \rightsquigarrow q([n] \circ a)$, for every n. If $f\colon a \to b$ is an arbitrary morphism in $\mathcal{C}$ then the diagram

$$\begin{array}{ccc} [n] \circ q(a) & \xrightarrow{l_{n|a}} & q([n] \circ a) \\ {\scriptstyle e_n \circ q(f)}\downarrow & & \downarrow{\scriptstyle q(e_n \circ f)} \\ [n] \circ q(b) & \xrightarrow[l_{n|b}]{} & q([n] \circ b) \end{array} \tag{3.53}$$

is commutative.

Proposition 3.18. *We have*

$$l_{m+n|a} = l_{m|[n]\circ a}(e_m \circ l_{n|a}) \tag{3.54}$$

for every $a \in \mathcal{C}$ and natural numbers m, n.

Proof. This follows easily from the above diagram (3.52), by accommodating two copies

$$\begin{array}{ccc} [m+n+k] \circ a & \xrightarrow{e_{m+n} \circ j_{k|a}} & [m+n] \circ q(a) \\ {\scriptstyle e_m \cdot \sigma_{nk} \circ i_a}\downarrow & & \downarrow{\scriptstyle e_m \circ l_{n|a}} \\ [m+k+n] \circ a & \xrightarrow[e_m \circ j_{k|[n]\circ a}]{} & [m] \circ q([n] \circ a) \\ {\scriptstyle \sigma_{mk} \circ i_{[n]\circ a}}\downarrow & & \downarrow{\scriptstyle l_{m|[n]\circ a}} \\ [k+m+n] \circ a & \xrightarrow[j_{k|[m+n]\circ a}]{} & q([m+n] \circ a) \end{array} \tag{3.55}$$

and observing that the outermost square defines $l_{m+n|a}$, hence (3.54) holds. □

So we have a natural transformation between q^2 and q. In general, it will be far from something associative.

We can inductively extend our constructions, to multidimensional arrays involving k indices $n_1, \ldots, n_k$, with

$$c_{n_1 \ldots n_k} = [n_1 + \cdots + n_k] \circ c \tag{3.56}$$

with elementary arrows induced by the canonical inclusions $[n_i] \to [n_i + 1]$. The main diagonal produces a sequence

$$c \longrightarrow [k] \circ c \longrightarrow [2k] \circ c \longrightarrow [3k] \circ c \longrightarrow \cdots$$

which can be made equivalent to the (subsequence of indices divisible by k) of the canonical sequence for c, via the k-fold shuffling rearrangements. This gives natural isomorphisms

$$\mu_{k|c} \colon q^k(c) \leftrightarrow q(c) \tag{3.57}$$

which can be viewed as iterated basic isomorphism $\mu_c \colon q^2(c) \to q(c)$.

Permutations act as natural automorphism of this multidimensional array, and hence as symmetries of the functors $q^k(c)$.

It turns out that the principal properties of the limit functor q are best studied separately, by postulating a framework in which they hold independently of our basic motivation—the constructions involving a complete and continuous action of natural numbers over $\mathcal{C}$. The next section is dedicated to the study of this more general scenario.

4 Internal Categorical Formulation

So here we present an 'internal' formulation retaining all basic properties of the constructions we have been studing above. We shall extract the essentials, without requiring the action of a symmetry category $\mathcal{N}$ on a category $\mathcal{C}$. We shall start with an abstract category $\mathcal{C}$ and gradually introduce the appropriate structure.

First of all, we shall assume that $\mathcal{C}$ is equipped with a covariant functor $q \colon \mathcal{C} \to \mathcal{C}$ and a natural transformations ι from the identity functor on $\mathcal{C}$ and q.

Since ι is a natural transformation, for every object $c \in \mathcal{C}$ we have a morphism $\iota_c \colon c \to q(c)$ such that the diagrams

$$\begin{array}{ccc} a & \xrightarrow{\iota_a} & q(a) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q(f)} \\ b & \xrightarrow{\iota_b} & q(b) \end{array} \tag{4.1}$$

are commutative, for arbitrary objects a and b and morphism $f\colon a \to b$. In particular, if we take $a = c$ and $b = q(c)$, with $f = \iota_c$, it follows that

$$\iota_{q(c)}\iota_c = q(\iota_c)\iota_c \tag{4.2}$$

a kind of 'coassociativity' property for morphisms ι_c. In other words, two natural equivalences between the identity functor and q^2 coincide. We shall denote by ι^2 this equivalence. In general, for every $k \geq 2$ we can construct in a natural way k equivalences between q^k and q^{k+1}. They are

$$\iota_{q^k(c)} \qquad q(\iota_{q^{k-1}(c)}) \qquad \cdots \qquad q^{k-1}(\iota_{q(c)}) \qquad q^k(\iota_c).$$

By sequentially combining these, we can construct equivalences ι^n between the identity functor and q^n. Due to the coassociativity these objects are independent of the choices of the above sequential equivalences. If we, on the other hand, start from q and apply sequential equivalences, we shall arrive on n in general different equivalences between q and q^n. In general, if $m < n$ there will be $\binom{n}{m}$ equivalences between functors q^m and q^n all naturally expressible in terms of ι and q. They are naturally labeled by monotone inclusion maps from m to n.

In particular, for $n = 2$ we have

$$\iota_{q(c)}, q(\iota_c)\colon q(c) \to q^2(c).$$

Our context is quite general, so we shall introduce two non-trivial restrictions.

Definition 4.1. We shall assume that the the objects $q^2(c)$ are generated by the above natural morphisms: If $f, g\colon q^2(c) \to d$ are morphisms such that

$$f\iota_{q(c)} = g\iota_{q(c)} \qquad fq(\iota_c) = gq(\iota_c)$$

then $f = g$. We shall call this condition a *minimality condition* for q. Furthermore, we shall assume that there exist morphisms $\sigma_c\colon q^2(c) \to q^2(c)$ such that

$$\sigma_c\iota_{q(c)} = q(\iota_c) \qquad \sigma_c q(\iota_c) = \iota_{q(c)} \tag{4.3}$$

for each $c \in \mathcal{C}$. This will be called a *transposability condition* for q and ι.

Let us observe that the minimality condition implies the uniqueness of the flip morphisms σ_c.

Remark 4.2. In general, a category $\mathcal{C}$ with ι and q will satisfy neither the minimality nor transposability. As an example of a non-minimal situation, we can consider the category of unitary spaces and linear maps, let q be a left tensor multiplication functor by a given space V of dimension ≥ 2, and define ι_c to be the inclusion $c \to \psi \otimes c$ where ψ is a given unitary vector of V. On the other hand, if $\mathcal{C}$ is a category in which every object possesses no non-trivial morphisms into itself, and if $\iota_{q(c)}$ and $q(\iota_c)$ are different morphisms, then certainly there can not exist any 'twist' σ_c between them. A simple realization is the category of natural numbers and monotone maps between them. In the next section, we shall study several quantum-geometric realizations of a minimal and transposable structure.

From now on, let us assume that the minimality and transposability properties hold.

Proposition 4.1. *For every $a, b \in \mathcal{C}$ and $f\colon a \to b$, the diagram*

$$\begin{array}{ccc} q^2(a) & \xrightarrow{\sigma_a} & q^2(a) \\ {\scriptstyle q^2(f)}\downarrow & & \downarrow{\scriptstyle q^2(f)} \\ q^2(b) & \xrightarrow[\sigma_b]{} & q^2(b) \end{array} \tag{4.4}$$

is commutative. In other words, all twists constitute a natural transformation $\sigma\colon q^2 \to q^2$. Moreover we have

$$\sigma_c = \sigma_c^{-1} \qquad \sigma_{q(c)} q(\sigma_c) \sigma_{q(c)} = q(\sigma_c) \sigma_{q(c)} q(\sigma_c). \tag{4.5}$$

Proof. Let us 'compose' the diagram (4.4) with $\iota_{q(a)}$ and $q(\iota_a)$. We compute

$$\begin{aligned} \sigma_b q^2(f) \iota_{q(a)} = \sigma_b \iota_{q(b)} q(f) = q(\iota_b) q(f) = q(\iota_b f) = q(q(f)\iota_a) \\ = q^2(f) q(\iota_a) = q^2(f) \sigma_a \iota_{q(a)} \end{aligned}$$

and similarly

$$q^2(f)\sigma_a q(\iota_a) = q^2(f)\iota_{q(a)} = \iota_{q(b)}q(f) = \sigma_b q(\iota_b)q(f) = \sigma_b q(\iota_b f)$$
$$= \sigma_b q(q(f)\iota_a) = \sigma_b q^2(f)q(\iota_a)$$

and hence by minimality (4.4) holds. The involutivity of σ is a direct consequence of the minimality as well. To prove the braid equation, let us use the same strategy, evaluating it on $\iota_{q^2(c)}$ and $q(\iota_{q(c)})$ respectively. We have

$$\sigma_{q(c)}q(\sigma_c)\sigma_{q(c)}\iota_{q^2(c)} = \sigma_{q(c)}q(\sigma_c)q(\iota_{q(c)}) = \sigma_{q(c)}q\big(\sigma_c\iota_{q(c)}\big) = \sigma_{q(c)}q^2(\iota_c) =$$
$$= q^2(\iota_c)\sigma_c = q\big(\sigma_c\iota_{q(c)}\big)\sigma_c =$$
$$= q(\sigma_c)q(\iota_{q(c)})\sigma_c = q(\sigma_c)\sigma_{q(c)}\iota_{q^2(c)}\sigma_c = q(\sigma_c)\sigma_{q(c)}q(\sigma_c)\iota_{q^2(c)}$$

and in the same spirit

$$\sigma_{q(c)}q(\sigma_c)\sigma_{q(c)}q(\iota_{q(c)}) = \sigma_{q(c)}q(\sigma_c)\iota_{q^2(c)} = \sigma_{q(c)}\iota_{q^2(c)}\sigma_c =$$
$$= q(\iota_{q(c)})\sigma_c = q\big(\sigma_c q(\iota_c)\big)\sigma_c = q(\sigma_c)q^2(\iota_c)\sigma_c = q(\sigma_c)\sigma_{q(c)}q^2(\iota_c) =$$
$$= q(\sigma_c)\sigma_{q(c)}q\big(\sigma_c\iota_{q(c)}\big) = q(\sigma_c)\sigma_{q(c)}q(\sigma_c)q(\iota_{q(c)}).$$

Thus, the braid equation holds & the proof is complete. □

Remark 4.3. Let us observe that the following generalization of the minimality condition can be established, for functors q^n where $n \geq 2$ is arbitrary. For $f, g\colon q^n(c) \to d$ we can conclude that $f = g$ if the composition of any of n natural morphisms $j\colon q(c) \to q^n(c)$ with f and g coincide. This can be easily derived inductively. For $n = 3$ the morphisms in question are

$$\iota_{q^2(c)}q(\iota_c) = q^2(\iota_c)\iota_{q(c)} \quad \iota_{q^2(c)}\iota_{q(c)} = q(\iota_{q(c)})\iota_{q(c)} \quad q^2(\iota_c)q(\iota_c) = q(\iota_{q(c)})q(\iota_c).$$

Remark 4.4. The inclusion morphisms together with twists realize a natural action of the category $\mathcal{N}$ on $\mathcal{C}$. We have

$$n \circ c = q^n(c) \tag{4.6}$$

for every $n \geq 0$. Because of the involutivity and the braid equations, for $n \geq 2$ then the twists acting on $q^n(c)$ realize the representation of the

permutation group S_n. Indeed, the elementary transpositions for $q^n(c)$ are given by

$$\sigma_{q^{n-2}(c)} \quad q(\sigma_{q^{n-3}(c)}) \quad \cdots \quad q^{n-3}(\sigma_{q(c)}) \quad q^{n-2}(\sigma_c).$$

It is easy to see that the non-neighbouring operators commute, and we have thus a representation of the permutation group.

Definition 4.5. Now we shall add a further postulate to our list of properties. We shall assume that there exists a functorial equivalence μ between q^2 and q satisfying

$$\mu_c \iota_c^2 = \iota_c. \tag{4.7}$$

This will be called the *infinity condition.*

So in summary, we have a category $\mathcal{C}$, a covariant functor q, and natural transformations ι, σ and μ. We assume that minimality, transposability and infinity properties hold.

Now we shall introduce the next fundamental structural element: a notion of 'local, gauge' transformations, acting on elements of the form $q(a)$. These symmetries will be instrumental in defining a new category, with much more freedom of transforming objects.

So we shall assume that for every $a \in \mathcal{C}$ a group $\mathcal{G}_a$ of automorphisms of $q(a)$ is given. These groups should respect, in a certain meaningful way, the structure of the category $\mathcal{C}$ with functor q and associated transformations.

At first, we shall additionally assume that for every $f\colon a \to b$ and every $u \in \mathcal{G}_a$ there exists $v \in \mathcal{G}_b$ such that the diagram

$$\begin{array}{ccc} q(a) & \xrightarrow{u} & q(a) \\ {\scriptstyle q(f)}\downarrow & & \downarrow{\scriptstyle q(f)} \\ q(b) & \xrightarrow[v]{} & q(b) \end{array} \tag{4.8}$$

is commutative. This is a kind of 'transportability' for internal symmetries– so that we can always move them along the direction of morphisms $q(f)$. It is worth noticing that if $q(f)$ is an isomorphism, then v is uniquely determined by u, and vice versa, as we have simply $v = q(f)uq(f)^{-1}$. In this case $\mathcal{G}_a$ and $\mathcal{G}_b$ will be the same.

Our second and third structural condition for symmetries, is that they are compatible with q–we ask that

$$\begin{aligned} u \in \mathcal{G}_c \quad &\Rightarrow \quad \mu_c q(u) \mu_c^{-1} \in \mathcal{G}_c \\ v \in \mathcal{G}_{q(c)} \quad &\Rightarrow \quad \mu_c v \mu_c^{-1} \in \mathcal{G}_c \end{aligned} \tag{4.9}$$

for every object $c \in \mathcal{C}$.

Our fourth and last condition for symmetries guarantees non-triviality of gauge symmetries, providing that certain previously introduced morphisms are non-trivial. Namely, we shall assume that

$$\tau_c = \mu_c \sigma_c \mu_c^{-1} \in \mathcal{G}_c \qquad \mathcal{G}_c \ni \omega_c = \mu_c q(\mu_c) \mu_{q(c)}^{-1} \mu_c^{-1}. \tag{4.10}$$

We always have a *minimal* solution, which consists of the groups $\mathcal{G}_c$ generated by initial transformations τ_c and ω_c and the following elementary operations

$$u_c \rightsquigarrow \mu_c q(u_c) \mu_c^{-1} \qquad u_c \rightsquigarrow \mu_c u_{q(c)} \mu_c^{-1}. \tag{4.11}$$

It is important to observe that all such transformations will be 'local' realizations of natural automophisms of q. We start with τ and ω and construct, by iteratively applying the above two operations, the set $\mathcal{S}$ of generating natural transformations. We then generate a group $\mathcal{G}$ from the set $\mathcal{S}$. The groups $\mathcal{G}_c$ will be simply realizations u_c for $u \in \mathcal{G}$.

It is instructive to verify, as an independent exercise, that the described symmetry groups indeed satisfy the 3 additional structural conditions. We can put it in a slightly more general context, by considering any initial collection Ω of natural transformations of q, and for consistency reasons we shall assume that $\Omega \ni \omega$. Then we proceed to construct the set $\mathcal{S} = \mathcal{S}^\Omega$. The setup above is given by $\Omega = \{\omega, \tau\}$. But we can consider also $\Omega = \{\omega\}$ the absolutely minimal context. We shall also write $\mathcal{S}^\omega$ and $\mathcal{S}^{\omega,\tau}$ in these particular cases, omitting the set parentheses. We shall also similarly write $\mathcal{G}^\Omega$ and in particular $\mathcal{G}^\tau$ and $\mathcal{G}^\omega$.

The transportability condition immediately follows from the naturality of our transformations. To verify the second and third conditions, let us first observe that they hold, by construction, for the elements of $\mathcal{S}$. In other words, if $u \in \mathcal{S}$ then $\mu\{qu\}\mu^{-1}$ and $\mu\{uq\}\mu^{-1}$ both belong to $\mathcal{S}$. And since

$$q[uv] = [qu][qv] \quad [uv]q = [uq][vq] \quad [uq]^{-1} = [u^{-1}]q \quad [qv]^{-1} = q[v^{-1}]$$

we conclude that the same property extends to $\mathcal{G}$. The following lemma gives an explicit description of the generating transformations.

Lemma 4.2. *The generating set $\mathcal{S}^{\Omega}$ of natural transformations is given by*

$$\mathcal{S}^{\Omega} = \bigcup_{k,l,F} F\big[q^k\{\Omega\}q^l\big]F^{-1} \tag{4.12}$$

where $n = k + l$ and F: $q^{n+2} \to q$ is any of sequential equivalences induced by combining q and μ.

Proof. The union of the transformations $F\big[q^k\{\Omega\}q^l\big]F^{-1}$ gives, by construction, a set closed under operations (4.11). On the other hand, all these transformations must belong to $\mathcal{S}^{\Omega}$, since they are obtained by iteratively applying the elementary operations on Ω. □

Lemma 4.3. *For any two natural transformations F and G between q^n and q where $n \geq 3$, obtained by sequentially combining q and μ, we have*

$$FG^{-1} \in \mathcal{G}^{\omega}. \tag{4.13}$$

In other words, one can always be obtained from another by composing with an element of the group $\mathcal{G}^{\omega}$.

Proof. We can prove it by induction on n. For $n = 3$ we have only two equivalences between q^3 and q, acting as $\mu_c q(\mu_c)$ and $\mu_c \mu_{q(c)}$. They are connected via ω_c. Now let us assume that the statement holds for some n and fix an equivalence E between q^n and q. Let us consider 'neighbouring' transformations from q^{n+1} to q, of the form

$$F_c = E_c q^k(\mu_{q^{l+1}c}) \qquad G_c = E_c q^{k+1}(\mu_{q^l(c)})$$

where $k + l = n + 1$. We shall prove that two such neighbours are always connectable via some $u \in \mathcal{G}^{\omega}$. By our induction hypothesis, we can replace E by an equivalent one, so that it assumes the following form

$$E_c = D_c q^k(\mu_c) \cdots q^k(\mu_{q^l(c)})$$

where D_c is a sequential equivalence between q^k and q, and if $k = 0$ it is simply the identity on q. Now we have

$$G_c = D_c q^k\{\mu_c \cdots \mu_{q^l(c)} q(\mu_{q^l(c)})\} = D_c q^k\{\mu_c \cdots \mu_{q^{l-1}(c)} \omega_{q^l(c)} \mu_{q^l(c)} \mu_{q^{l+1}(c)}\}$$
$$= u_c D_c q^k\{\mu_c \cdots \mu_{q^{l-1}(c)} \mu_{q^l(c)} \mu_{q^{l+1}(c)}\} = u_c E_c q^k(\mu_{q^{l+1}(c)}) = u_c F_c$$

where

$$u_c = D_c q^k\{\mu_c \cdots \mu_{q^{l-1}(c)}\} q^k(\omega_{q^l(c)}) q^k\{\mu_c \cdots \mu_{q^{l-1}(c)}\}^{-1} D_c^{-1}$$

is clearly an element of $\mathcal{G}^\omega$. □

Remark 4.6. Let $\mathcal{G}$ be the group of *all natural automorphisms* of the functor q. By definition the elements $g \in \mathcal{G}$ are realizable as $g_c\colon q(c) \to q(c)$ such that the diagrams

$$\begin{array}{ccc} q(a) & \xrightarrow{g_a} & q(a) \\ {\scriptstyle q(f)}\downarrow & & \downarrow{\scriptstyle q(f)} \\ q(b) & \xrightarrow{g_b} & q(b) \end{array} \tag{4.14}$$

are commutative, this expresses the naturality of $g \in \mathcal{G}$.

Remark 4.7. Another interesting condition for transformations $u \in \mathcal{G}_c$ is the preservation of morphisms ι_c, in the sense of $u\iota_c = \iota_c$. It is easy to see that our both minimal solutions satisfy this property.

The transportability conditions tell us that morphisms $q(f)\colon q(a) \to q(b)$ behave in a special way relative to the symmetry transformations from the groups $\mathcal{G}_a$ and $\mathcal{G}_b$. They are *conductive* to the symmetries, so to speak. It is interesting to consider other possible morphisms possessing the same property. This motivates our next definition.

Definition 4.8. A morphism $F\colon q(a) \to q(b)$ is called *conductive* (relative to a and b) iff for every $u \in \mathcal{G}_a$ there exists $v \in \mathcal{G}_b$ such that $Fu = vF$. Obviously all morphisms of the form $q(f)$ where $f\colon a \to b$ are conductive.

As a direct application of this definition, we see that if $F\colon q(a) \to q(b)$ and $G\colon q(b) \to q(c)$ are conductive, then so is the composition $GF\colon q(a) \to q(c)$. In what follows, we shall denote by $\Phi(f)$ the expression $\mu_b q(f)$ for a given $f :: a \to q(b)$.

Lemma 4.4. *For every morphism $f\colon a \to q(b)$ the morphism $\Phi(f)\colon q(a) \to q(b)$ is conductive. Explicitly*

$$\Phi(f)u = \mu_b v \mu_a^{-1} \Phi(f) \tag{4.15}$$

where $v \in \mathcal{G}_{q(b)}$ is a symmetry satisfying $vq(f) = q(f)u$.

Proof. We have

$$\Phi(f)u = \mu_b q(f) u = \mu_b v q(f) = \mu_b v \mu_b^{-1} \Phi(f)$$

and let us observe that $\mu_b v \mu_b^{-1}$ belongs to $\mathcal{G}_b$ according to the second structural condition for symmetries. □

Proposition 4.5. *For every $g \in \mathcal{G}$ the natural tranformations*

$$u = \mu\{hg\}\mu^{-1} \qquad v = \mu\{gh\}\mu^{-1} \tag{4.16}$$

again belong to $\mathcal{G}$.

Proof. It is clear that u and v defined above are natural transformations of the functor q into itself. The morphisms u_c and v_c are given by

$$\begin{array}{ccc} q^2(a) & \xrightarrow[q(g_c)]{g_{q(c)}} & q^2(a) \\ {\scriptstyle \mu_c}\downarrow & & \downarrow{\scriptstyle \mu_c} \\ q(a) & \xrightarrow[v_c]{u_c} & q(a) \end{array} \tag{4.17}$$

A simple use of the identities

$$\mu_{q(c)} \iota^2_{q(c)} = \iota_{q(c)} \qquad q(\mu_c) q(\iota_c^2) = q(\iota_c) \tag{4.18}$$

which are a direct consequence of the coassociativity of ι applied to $q(c)$ and 'acted on' by q respectively, leads us to conclude that ι_c is invariant under u_c and v_c. □

The twist σ naturally generate a transformation from $\mathcal{G}$. Namely, if we define

$$s_c = \mu_c \sigma_c \mu_c^{-1} \tag{4.19}$$

then it is easy to see that s_c is involutive automorphism of $q(c)$ preserving the transformation ι_c.

The defining property of the transformation μ realizing equivalence between q^2 and q is that it intertwines ι^2 and ι. Any other such an equivalence say μ^* will perform equally well as μ.

Remark 4.9. It is clear that the two equivalences are always related by a q-automorphism from $\mathcal{G}$, so that we have $\mu_c^* = g_c \mu_c$.

Lemma 4.6. *Let us consider the 2 natural equivalences between q^3 and q given by $\mu_{q(c)}, q(\mu_c) \colon q^3(c) \to q(c)$ There exists a unique $g \in \mathcal{G}$ such that*

$$g_c \mu_c q(\mu_c) = \mu_c \mu_{q(c)}. \tag{4.20}$$

In other words, although μ is not in general associative, it is 'almost' associative.

Proof. For a given $n \geq 3$ we can construct the following natural equivalences between q^n and q^{n-1}, in general they will all be different transformations:

$$\mu_{q^{n-2}(c)} \quad q(\mu_{q^{n-3}(c)}) \quad \cdots \quad q^{n-3}(\mu_{q(c)}) \quad q^{n-2}(\mu_c).$$

By sequentially combining such transformations, we can construct equivalences of the type $\tau \colon q^n \to q$. Now as a consequence of the basic property (4.7) we have

$$\tau_c \iota_c^n = \iota_c \tag{4.21}$$

for every object c of $\mathcal{C}$. □

It is worth mentioning that these intertwining properties are all special cases of a general phenomenon.

Proposition 4.7. *Let us consider any equivalence τ between functors q^n and q^m, constructed by combining σ, μ, ι_c and q. Such an equivalence always intertwines ι^n and ι^m.*

Proof. The following are the basic structural elements which produce general expressions in the sense of taking compositions of morphisms:

$$q^k[\sigma_{q^l(c)}]\colon q^{k+l+2}(c) \to q^{k+l+2}(c) \qquad q^k[\mu_{q^l(c)}]\colon q^{k+l+2}(c) \to q^{k+l+1}(c)$$
$$q^k[\iota_{q^l(c)}]\colon q^{k+l}(c) \to q^{k+l+1}(c).$$

It is sufficient to check that these maps satisfy the stated intertwinning property. □

We shall now describe a natural extension of the category $\mathcal{C}$, containing the same objects as $\mathcal{C}$, but having (in general) substantially richer morphisms base. Our constructions can be viewed as generalizations and extensions of the constructions discussed previously. For given objects $a, b \in \mathcal{C}$ let us begin by defining *new morphisms* between them.

Definition 4.10. We shall say that morphisms $f, g\colon a \to q(b)$ are *equivalent* and write $f \sim_q g$ iff there exist $u \in \mathcal{G}_b$ such that $uf = g$. It is clear that the this relation is an equivalence relation. In fact, its classes are orbits relative to the natural action of $\mathcal{G}_b$ on morphisms from a to $q(b)$.

Lemma 4.8. *Let us consider transformations*

$$\Pi_c^+ = \mu_c \iota_{q(c)} \qquad \Pi_c^- = \mu_c q(\iota_c) \tag{4.22}$$

of $q(c)$. They are mutually symmetric via τ_c in the sense

$$\tau_c \Pi_c^- = \Pi_c^+ \qquad \tau_c \Pi_c^+ = \Pi_c^-. \tag{4.23}$$

Moreover, these transformations are idempotent, modulo the action of the gauge symmetries of $\mathcal{G}_c$: we have

$$\Pi_c^+ \Pi_c^+ = \omega_c^{-1} \Pi_c^+ \qquad \Pi_c^- \Pi_c^- = \omega_c \Pi_c^- \tag{4.24}$$

for every object $c \in \mathcal{C}$.

Proof. Equation (4.23) is a direct consequence of the intertwining & defining property of σ_c, and the definition of τ_c. Let us calculate the square of the introduced operators. We have

$$\mu_c \iota_{q(c)} \mu_c \iota_{q(c)} = \mu_c q(\mu_c) \iota_{q^2(c)} \iota_{q(c)} = \mu_c q(\mu_c) \iota_{q(c)}^2 =$$
$$= \mu_c q(\mu_c) \iota_{q(c)}^2 = \omega_c \mu_c \mu_{q(c)} \iota_{q(c)}^2 = \omega_c \mu_c \iota_{q(c)}.$$

Similarly

$$\mu_c q(\iota_c)\mu_c q(\iota_c) = \mu_c \mu_{q(c)} q^2(\iota_c) q(\iota_c) = \mu_c \mu_{q(c)} q(\iota_c^2)$$
$$== \omega_c^{-1} \mu_c q(\mu_c) q(\iota_c^2) = \omega_c^{-1} \mu_c q\big(\mu_c \iota_c^2\big) = \omega_c^{-1} \mu_c q(\iota_c)$$

and the proof is complete. □

These almost-idempotents interface quite well with the group $\mathcal{G}$. If $g \in \mathcal{G}$ is an arbitrary automorphism of q then

$$\Pi_c^- g_c = \{\mu_c q(\iota_c)\} g_c = \mu_c g_{q(c)} q(\iota_c) = \{\mu_c g_{q(c)} \mu_c^{-1}\}\{\mu_c q(\iota_c)\} = \{\mu_c g_{q(c)} \mu_c^{-1}\}\Pi_c^-$$
$$\Pi_c^+ g_c = \{\mu_c \iota_{q(c)}\} g_c = \mu_c q(g_c) \iota_{q(c)} = \{\mu_c q(g_c) \mu_c^{-1}\}\{\mu_c \iota_{q(c)}\} = \{\mu_c q(g_c) \mu_c^{-1}\}\Pi_c^+.$$

Lemma 4.9. *If $f, g\colon a \to q(b)$ are q-equivalent morphisms then the morphisms $\Pi_c^- f$ and $\Pi_c^- g$ are q-equivalent.*

Proof. It follows directly from the above relations. We could have also used the equivalent formulation in terms of $\mu_c q(\iota_c)$. □

Thus we have constructed a natural idempotent, acting in the space of equivalence classes of morphisms from a to $q(b)$. We shall be mainly interested in classes that belong to the image of this projector (in other words, the invariant classes).

Definition 4.11. A morphism $f\colon a \to q(b)$ is called *balanced* iff

$$\{\mu_b \iota_{q(b)}\} f \sim_q f \quad \Leftrightarrow \quad f \sim_q \{\mu_b q(\iota_b)\} f. \tag{4.25}$$

In other words, its equivalence class does not change after composing with the canonical projector.

Clearly if f is balanced and $g \sim_q f$ then g is balanced too. This follows from the above, but we can also verify it explicitly

$$\{\mu_b \iota_{q(b)}\} g = \{\mu_b \iota_{q(b)}\} u_b f = \{\mu_b u_{q(b)} \mu_b^{-1}\}\{\mu_b \iota_{q(b)}\} f = \{\mu_b u_{q(b)} \mu_b^{-1}\} v_b f =$$
$$= \{\mu_b u_{q(b)} \mu_b^{-1}\} v_b u_b^{-1} g.$$

It is worth observing that

$$\Pi_c^- \iota_c = \iota_c \tag{4.26}$$

so the morphisms $\iota_c\colon c \to q(c)$ are always balanced.

Lemma 4.10. *We have*

$$\mu_c q(\Pi_c^-)\mu_c^{-1} \sim_q \Pi_c^- \sim_q \mu_c \Pi_{q(c)}^- \mu_c^{-1} \tag{4.27}$$

for every $c \in \mathcal{C}$.

Proof. A direct calculation gives

$$\mu_c q(\Pi_c^-)\mu_c^{-1} = \mu_c q(\mu_c) q(\iota_{q(c)})\mu_c^{-1} = \mu_c q(\mu_c)\sigma_{q(c)}\iota_{q^2(c)}\mu_c^{-1} = \\ = \mu_c q(\mu_c)\sigma_{q(c)} q(\mu_c^{-1})\iota_{q(c)} = \{\mu_c q(\mu_c)\sigma_{q(c)} q(\mu_c)^{-1}\mu_c^{-1}\}\Pi_c^-$$

which proves the first equivalence. Similarly,

$$\mu_c \Pi_{q(c)}^- \mu_c^{-1} = \mu_c \mu_{q(c)}\iota_{q^2(c)}\mu_c^{-1} = \mu_c \mu_{q(c)} q(\mu_c^{-1})\iota_{q(c)} = \\ = \{\mu_c \mu_{q(c)} q(\mu_c)^{-1}\mu_c^{-1}\}\Pi_c^- = \omega_c \Pi_c^-$$

and the second equivalence thus holds. Let us recall that all algebraic expressions involving ι, μ_c and σ, are always intertwining the canonical imbeddings. In particular the natural transformation $\mu_c q(\mu_c)\sigma_{q(c)} q(\mu_c)^{-1}\mu_c^{-1}$ belongs to $\mathcal{G}$. □

It is worth observing that similar transformation properties apply for Π_c^+. We have

$$\begin{aligned} \mu_c q(\Pi_c^+)\mu_c^{-1} &= \omega_c \Pi_c^+ \\ \mu_c \Pi_{q(c)}^+ \mu_c^{-1} &= \{\mu_c \mu_{q(c)}\sigma_{q(c)}\mu_{q(c)}^{-1}\mu_c^{-1}\}\Pi_c^+ \end{aligned} \tag{4.28}$$

as it easily follows from the same type of arguments.

Lemma 4.11. *If* $f\colon a \to q(b)$ *is balanced, then* $\mu_b q(f)\colon q(a) \to q(b)$ *is a balanced morphism too.*

Proof. We compute

$$\Pi_b^- \mu_b q(f) \sim_q \mu_b q(\Pi_b^-)\mu_b^{-1}\mu_b q(f) = \mu_b q\big(\Pi_b^- f\big) = \{\mu_b q(u_b)\mu_b^{-1}\}\mu_b q(f)$$

assuming that $\Pi_b^- f = u_b f$ for some $u \in \mathcal{G}$. This shows that $\mu_b q(f)$ is balanced. □

Lemma 4.12. *If $f, g\colon a \to q(b)$ are equivalent, then $\mu_b q(f)$ and $\mu_b q(g)$ are equivalent, too.*

Proof. Garlic and water:

$$\mu_b q(g) = \mu_b q(u_b) q(f) = \{\mu_b q(u_b)\mu_b^{-1}\}\mu_b q(f)$$

where we have assumed that $g = u_b f$. □

We proceed to define *the composition* of q-morphisms. For $\mathcal{C}$-morphisms $f\colon a \to q(b)$ and $g\colon b \to q(c)$ let us consider the composition $f \diamond g\colon a \to q(c)$ given by

$$f \diamond g = \mu_c q(g) f. \tag{4.29}$$

In general such a composition operator would not be associative. However, it turns out that it is compatible with the notion of balanceness, factorizable over $\sim_q$, and such a factorized $\diamond$ is indeed associative. Establishing this is our next immediate goal.

Lemma 4.13. *If $f\colon a \to q(b)$ is balanced and $g\colon c \to q(a)$ is arbitrary, then $g \diamond f$ is balanced.*

Proof. This is a direct consequence of Lemma 4.11 and the above definition of the composition. □

Lemma 4.14. *If $f, g\colon a \to q(b)$ are equivalent and $h\colon c \to q(a)$ is arbitrary, then $h \diamond f$ and $h \diamond g$ are equivalent.*

Proof. This is a direct consequence of Lemma 4.12 and the definition of the composition rule. □

Lemma 4.15. *If $f, g\colon a \to q(b)$ are equivalent and $h\colon b \to q(c)$ is arbitrary, then $f \diamond h$ and $g \diamond h$ are equivalent.*

Proof. Assuming that $g = u_b f$ then

$$g \diamond h = \mu_c q(h) g = \mu_c q(h) u_b f = \mu_c u_{q(c)} q(h) f = \{\mu_c u_{q(c)} \mu_c^{-1}\}\mu_c q(h) f = \\ = \{\mu_c u_{q(c)} \mu_c^{-1}\}\{f \diamond h\}$$

in other words $f \diamond h$ and $g \diamond h$ are equivalent. □

The last two lemmas tell us that the composition $\diamond$ is compatible with $\sim_q$. Therefore we can consistently define the composition at the level of equivalence classes simply by projecting

$$[f] \diamond [g] = [f \diamond g] \tag{4.30}$$

down to the factor space. According to Lemma 4.11 we can also restrict ourselves to balanced morphisms/classes, and the composition will always be balanced.

Lemma 4.16. *For arbitrary morphisms $f\colon a \to q(b)$, $g\colon b \to q(c)$ and $h\colon c \to q(d)$ we have*

$$(f \diamond g) \diamond h = \omega_d\big\{f \diamond (g \diamond h)\big\}. \tag{4.31}$$

In particular, this means that the factorized composition is associative.

Proof. We compute

$$(f \diamond g) \diamond h = \big(\mu_c q(b) f\big) \diamond h = \mu_d q(h) \mu_c q(g) f = \mu_d \mu_{q(d)} q^2(h) q(g) f =$$
$$\omega_d \mu_d q(\mu_d) q^2(h) q(g) f = \omega_d \mu_d q\big(\mu_d q(h) g\big) f = \omega_d \mu_d q\big(g \diamond h\big) f = \omega_d\big\{f \diamond (g \diamond h)\big\}$$

and thus, the stated equivalence property holds. □

Lemma 4.17. *For every $f\colon a \to q(b)$ we have*

$$\iota_a \diamond f = \Pi_b^- f \qquad f \diamond \iota_b = \tau_b \Pi_b^- f. \tag{4.32}$$

In particular, if f is balanced then $\iota_a \diamond f$ and $f \diamond \iota_b$ are equivalent to f.

Proof. We have $\iota_a \diamond f = \mu_b q(f) \iota_a = \mu_b \iota_{q(b)} f = \Pi_b^- f$ and similarly $f \diamond \iota_b = \mu_b q(\iota_b) f = \mu_b \sigma_b \mu_b^{-1} \Pi_b^- f = \tau_b \Pi_b^- f$ and thus the statement. □

Definition 4.12. An q-morphism between a and b is an equivalence class of balanced $\mathcal{C}$-morphisms between a and $q(b)$.

Proposition 4.18. *If we endow the objects of $\mathcal{C}$ with these new morphisms, and their composition $\diamond$, we obtain a new category $\mathcal{C}^*$. The neutral morphism of $a \in \mathcal{C}^*$ is given by the class of ι_c. There exists a canonical functor $\zeta\colon \mathcal{C} \to \mathcal{C}^*$, acting trivially on objects and transforming each $f\colon a \to b$ into the class of $\iota_b f$.*

Proof. We have already established necessary results for the proof. Let us here check that ζ is indeed a covariant functor. For every object c of $\mathcal{C}$, it tranforms i_c into $[\iota_c]$, which is the identity morphism of c in $\mathcal{C}^*$. If $f\colon a \to b$ and $g\colon b \to c$ are morphisms in $\mathcal{C}$ then

$$[\zeta(f)] \diamond [\zeta(g)] = [\zeta(g) \diamond \zeta(g)] = [(\iota_b f) \diamond (\iota_c g)] = [\mu_c q(\iota_c g)\iota_b f] = \\ = [\mu_c q(\iota_c) q(g) \iota_b f] = [\mu_c q(\iota_c)\iota_c g f] = [\iota_c g f] = \zeta(gf)$$

and hence ζ is indeed a covariant functor. □

There is another, equivalent, description of the category $\mathcal{C}^*$ in which the domain and codomain of morphisms are symmetrically represented.

Let us consider objects $a, b \in \mathcal{C}$ and morphisms between a and $q(b)$ as well as between $q(a)$ and $q(b)$. As we have already mentioned, for $f\colon a \to q(b)$, we can construct $\psi = \Phi(f)$ which is a morphism between $q(a)$ and $q(b)$. And for an arbitrary $\psi\colon q(a) \to q(b)$, we can construct $f = \psi\iota_a\colon a \to q(b)$.

These operations are almost inverse to each other. We shall now prove a series of at a first sight technical properties connecting these two types of morphisms, and involving a couple of other interesting things.

Lemma 4.19. *The following identity holds*

$$\Pi_b^- F = \{\mu_b q(F) \mu_a^{-1}\}\Pi_a^- \tag{4.33}$$

for arbitrary objects a, b in $\mathcal{C}$ and a morphism $F\colon q(a) \to q(b)$.

Proof. Elementary calculation

$$\Pi_b^- F = \mu_b \iota_{q(b)} F = \mu_b q(F) \iota_{q(a)} = \{\mu_b q(F)\mu_a^{-1}\}\Pi_a^-$$

and—voilà. □

Lemma 4.20. *If $\psi = \Phi(f)$ for an arbitrary $f\colon a \to q(b)$ then*

$$\mu_b q(\psi) \mu_a^{-1} = \omega_b \psi. \tag{4.34}$$

On the other hand, if $\psi\colon q(a) \to q(b)$ is given and we define $f = \psi\iota_a$ then

$$\Pi_b^- f = \{\mu_b q(\psi)\mu_a^{-1}\}\iota_a. \tag{4.35}$$

In particular if ψ satisfies (4.34) *then the associated f satisfies*

$$\Pi_b^- f = \omega_b f \tag{4.36}$$

and it follows that it is balanced.

Proof. We calculate

$$\mu_b q(\psi)\mu_a^{-1} = \mu_b q\big(\mu_b q(f)\big)\mu_a^{-1} = \mu_b q(\mu_b) q^2(f)\mu_a^{-1} = \omega_b \mu_b \mu_{q(b)} q^2(f)\mu_a^{-1} = \\ = \omega_b \mu_b q(f) = \omega_b \psi.$$

Now if ψ satisfies this property and we define $f = \psi\iota_a$ then

$$\Pi_b^- f = \mu_b \iota_{q(b)} \psi \iota_a = \mu_b q(\psi) \iota_{q(a)} \iota_a = \mu_b q(\psi)\mu_a^{-1}\iota_a$$

which completes the proof. □

Lemma 4.21. *If $f\colon a \to q(b)$ satisfies* (4.36) *then for $\psi = \Phi(f)$ we have*

$$\psi\Pi_a^+ = \omega_b^{-1}\{\mu_b q(\omega_b)\mu_a^{-1}\}\psi. \tag{4.37}$$

This property could be called a right balanceness condition for ψ.

Proof. If the mentioned property holds, then

$$\psi\Pi_a^+ = \mu_b q(f)\mu_a q(\iota_a) = \mu_b \mu_{q(b)} q^2(f) q(\iota_a) = \mu_b \mu_{q(b)} q\big(\iota_{q(b)} f\big) \\ = \mu_b \mu_{q(b)} q(\mu_b)^{-1} q\big(\Pi_b^+ f\big) = \omega_b^{-1}\{\mu_b q(\omega_b)\mu_b^{-1}\}\psi$$

and the proof is complete. □

As a corolary of the above two lemmas, we can formulate an important characterization.

Lemma 4.22. *A necessary and sufficient condition for a morphism ψ to be equivalent to a map of the form $\Phi(f)$ for some balanced $f\colon a \to q(b)$ is*

$$\mu_b q(\psi)\mu_a^{-1} \sim_q \psi \sim_q \psi\Pi_a^+. \tag{4.38}$$

In this case, we can define $f = \psi\iota_a$.

Proof. It is obtained by a simple combination of the properties deduced in the above two lemmas, and elementary complementary calculations. □

A simple but important fact worth observing here is that a morphism equivalent to a conductible morphism $F\colon q(a) \to q(b)$ is again conductible, and therefore all the morphisms equivalent to $\Phi(f)$ are conductible.

Lemma 4.23. *For arbitrary $\psi\colon q(a) \to q(b)$ and $f\colon a \to q(b)$ we have*

$$\begin{aligned} \Phi(\psi\iota_a) &= \psi\Pi_a^+ \\ \mu_b q(f)\iota_a &= \Pi_b^- f. \end{aligned} \tag{4.39}$$

The assignements

$$f \rightsquigarrow \mu_b q(f) \qquad \psi \rightsquigarrow \psi\iota_a \tag{4.40}$$

induce a natural one-to-one correspondence between classes of equivalence of balanced morphisms f and classes of equivalence of doubly balanced morphisms ψ, the double balanceness being defined as

$$\{\mu_b\iota_{q(b)}\}\psi \sim_q \psi \sim_q \psi\{\mu_a q(\iota_a)\}. \tag{4.41}$$

Proof. Identities (4.39) are easy exercise of elementary properties of μ and ι. If $\psi\colon q(b) \to q(b)$ is doubly balanced, then $f = \psi\iota_a$ is balanced, and we have $\mu_b q(f) \sim_q f$. On the other hand, if $f\colon a \to q(b)$ is arbitrary, then $\psi = \mu_b q(f)$ satisfies

$$\begin{aligned} \psi\Pi_a^+ &= \mu_b q(f)\mu_a q(\iota_a) = \mu_b\mu_{q(b)}q^2(f)q(\iota_a) = \mu_b\mu_{q(b)}q\big(q(f)\iota_a\big) \\ &= \mu_b\mu_{q(b)}q\big(\iota_{q(b)}f\big) = \mu_b\Pi_{q(b)}^+\mu_b^{-1}\psi = \{\mu_b\mu_{q(b)}\sigma_{q(b)}\mu_{q(b)}^{-1}\mu_b^{-1}\}\Pi_b^+\psi. \end{aligned}$$

Now if f is balanced, it follows that ψ is balanced, as we have already seen in Lemma 4.11. Moreover the above equality implies that composing Π_a^+ and ψ does not change its equivalence class. Hence ψ is doubly balanced. Also $\psi\iota_a \sim_q f$. It is clear that the introduced assignements preserve the equivalence relation. □

So we can indeed interpret morphisms between a and b in $\mathcal{C}^*$ as equivalence classes of doubly balanced morphisms between $q(a)$ and $q(b)$. In terms of such a representation, the composition is simply the old composition in $\mathcal{C}$. It is clear that the standard composition of morphisms is compatible with the introduced equivalence relation.

Lemma 4.24. *The introduced assignements intertwine the composition* $\diamond$ *and the standard composition of morphisms.*

Proof. If $f\colon a \to q(b)$ and $g\colon b \to q(c)$ are given, with $\psi = \mu_b q(f)$ and $\varphi = \mu_c q(g)$ then

$$\varphi\psi = \mu_c q(g)\mu_b q(f) = \mu_c \mu_{q(c)} q^2(g) q(f) = \omega_c \mu_c q(\mu_c) q^2(g) q(f) = \\ = \omega_c \mu_c q\big(\mu_c q(g) f\big) = \omega_c \mu_c q\big(f \diamond g\big)$$

and so the assignements indeed intertwine the 2 compositions. □

The functor will faithfully represent the morphisms of $\mathcal{C}$ iff all ι_c are monomorphisms in $\mathcal{C}$.

All underlying morphisms producing such a realization of $\mathcal{C}^*$ are conductible. Therefore we can introduce an intermediary category $\mathcal{D}$ having the same objects as $\mathcal{C}$, and where the morphisms between a and b are those morphisms $F\colon q(a) \to q(b)$ for which the morphism $\mu_b q(F)\mu_a^{-1}$ is conductible. So we obtain $\mathcal{C}^*$ from $\mathcal{D}$ by considering doubly-balanced morphisms and factorizing over the action of gauge symmetries.

Lemma 4.25. *If* $F\colon q(a) \to q(b)$ *is a morphism in* $\mathcal{D}$ *between* a *and* b *then* $\mu_b q(F)\mu_a^{-1}$ *belongs to* $\mathcal{D}$*-morphisms between* a *and* b *too.*

Proof. This is a direct consequence of the definition of $\mathcal{D}$ and formula

$$\mu bq\{\mu_b q(F)\mu_a^{-1}\}\mu_a^{-1} = \omega_b\{\mu_b q(F)\mu_a^{-1}\}\omega_a^{-1} \tag{4.42}$$

which holds for arbitrary $F\colon q(a) \to q(b)$. To prove the formula, we compute

$$\mu bq\{\mu_b q(F)\mu_a^{-1}\}\mu_a^{-1} = \mu bq(\mu_b q^2(F) q(\mu_a)^{-1}\mu_a^{-1} = \\ = \omega_b \mu_b \mu_{q(b)} q^2(F)\mu_{q(a)}^{-1}\mu_a^{-1}\omega_a^{-1} = \\ = \omega_b \mu_b q(F)\mu_{q(a)}\mu_{q(a)}^{-1}\mu_a^{-1}\omega_a^{-1} = \omega_b\{\mu_b q(F)\mu_a^{-1}\}\omega_a^{-1}.$$

Now if F is conductive, we can write

$$\{\mu_b q(F)\mu_a^{-1}\}\omega_a^{-1} = u\{\mu_b q(F)\mu_a^{-1}\} \sim_q \mu_b q(F)\mu_a^{-1}$$

and thus the statement follows. □

In general, being different categories, the notion of isomorphism will be different. However, being isomorphic in $\mathcal{D}$ implies being isomorphic in $\mathcal{C}^*$. And under some additional conditions, we can prove that isomorphisms in $\mathcal{C}^*$ are actually equivalent to isomorphisms in $\mathcal{D}$.

Lemma 4.26. *If objects a and b are $\mathcal{D}$-isomorphic then they are also $\mathcal{C}^*$-isomorphic.*

Proof. Let us consider mutually inverse morphisms $F\colon q(a) \to q(b)$ and $G\colon q(b) \to q(a)$ belonging to $\mathcal{D}$. In this case we can as well assume that

$$F = u\mu_b q(F)\mu_a^{-1} \qquad G = v\mu_a q(G)\mu_b^{-1}$$

since we can always make a shift

$$F \rightsquigarrow \mu_b q(F)\mu_a^{-1} \qquad G \rightsquigarrow \mu_a q(G)\mu_b^{-1}$$

and we apply Lemma 4.25. So, for such F and $G = F^{-1}$ we have

$$\mu_b q(F)\mu_a^{-1} = \omega_b F \omega_a^{-1} \qquad \mu_a q(G)\mu_b^{-1} = \omega_a G \omega_b^{-1}.$$

By applying (4.33) we find

$$\{\omega_b^{-1}\Pi_b^-\}F = F\{\omega_a^{-1}\Pi_a^-\} \quad \Leftrightarrow \quad G\{\omega_b^{-1}\Pi_b^-\} = \{\omega_a^{-1}\Pi_a^-\}G.$$

If we now consider the morphisms F and G composed with the projectors as above, and denote these maps by F' and G' then

$$F'G' = \omega_b^{-1}\Pi_b^- F \omega_a^{-1}\Pi_a^- G = \left(\omega_b - 1\Pi_b^-\right)^2 = \omega_b^{-1}\mu_b \iota_{q(b)}\omega_b^{-1}\Pi_b^- =$$
$$\omega_b^{-1}\mu_b q(\omega_b^{-1})\iota_{q(b)}\Pi_b^- = \omega_b^{-1}\mu_b q(\omega_b^{-1})\mu_b^{-1}\Pi_b^-\Pi_b^- = \omega_b^{-1}\mu_b q(\omega_b^{-1})\mu_b^{-1}\omega_b\Pi_b^-.$$

Similarly, we conclude that $G'F' = \omega_a^{-1}\mu_a q(\omega_a)^{-1})\mu_a^{-1}\omega_a\Pi_a^-$. On the other hand, as it is easy to see, both F' and G' are doubly balanced morphisms. Therefore, as representatives of morphisms in $\mathcal{C}^*$ they are mutually inverse, and hence providing an isomorphism between a and b in $\mathcal{C}^*$. □

Remark 4.13. A very important special case is when arbitrary isomorphisms between $q(a)$ and $q(b)$ belong to $\mathcal{D}$. Such a scenario is realized when for example objects of $\mathcal{C}$ are C*-algebras, with morphisms being unital *-homomorphisms, and the gauge transformations are given by inner automorphisms.

Remark 4.14. For a given object c, we can destilate an important subgroup, consisting of all those transformations of the form $v = \mu_c^{-1} q(u) \mu_c$. These are sufficient to express all equivalences, as we have

$$\Pi_c^- u = \{\mu_c q(u) \mu_u^{-1}\} \Pi_c^- . \tag{4.43}$$

Applying (4.42) in this context of gauge transformations gives us an explicit formula to calculate such 'normalized' objects.

Let us conclude this analysis by showing that the construction applied to the new category $\mathcal{C}^*$ will not produce anything really new. In other words, $\mathcal{C}^{**}$ is identical to $\mathcal{C}^*$. First of all, we have to show that the construction is applicable to $\mathcal{C}^*$. Let us define the functor q in this new context. Since the objects are the same as those of $\mathcal{C}$, we have to define the functor[†] q on morphisms of $\mathcal{C}^*$. By definition q will transform the class of $f\colon a \to q(b)$ into the class of $q(f)\colon q(a) \to q^2(b)$. Such a definition is consistent with the equivalence relation $\sim_q$ and balanceness. Indeed, we have

$$q(uf) = q(u)q(f) \qquad q(\Pi_b^-) = q(\mu_b)\mu_{q(b)}^{-1}$$

for every $u \in \mathcal{G}_b$. So our definition is consistent. The new q is a covariant functor, because

$$q\{f \cdot g\} = q(\mu_c)\mu_{q(c)}^{-1}\{q(f) \cdot q(g)\} \qquad q(\iota_c) = \sigma_c \iota_{q(c)}$$

so it tranforms the identity arrows into identity arrows and the composition in $\mathcal{C}^*$ of f and $g\colon b \to q(c)$ into the composition of $q(f)$ and $q(g)$.

Let us now verify that μ_c and ι_c viewed as morphisms of $\mathcal{C}^*$, provide natural transformations between q^2 and q, and the identity functor in $\mathcal{C}^*$ and q, respectively. Here we are interpreting morphisms in $\mathcal{C}$ as realized in $\mathcal{C}^*$ via the canonical functor $\mathcal{C} \rightsquigarrow \mathcal{C}^*$.

For a given $f\colon a \to q(b)$ we have

$$\begin{aligned} f \diamond [\iota_b] &= \mu_{q(b)} q(\iota_{q(b)}\iota_b) f = \mu_{q(b)} q(\mu_b)^{-1} q(\mu_b \iota_{q(b)} \iota_b) f = \mu_{q(b)} q(\mu_b)^{-1} q(\iota_b) f \\ &= \mu_{q(b)} q(\mu_b)^{-1} \sigma_b \iota_{q(b)} f = \mu_{q(b)} q(\mu_b^{-1}) \sigma_b q(f) \iota_a = \mu_{q(b)} q(\mu_b^{-1}) \sigma_b q(f) \mu_a \iota_{q(a)} \iota_a \\ &= \mu_{q(b)} q(\mu_b^{-1}) \sigma_b \mu_{q(b)} q^2(f) \iota_{q(a)} \iota_a = \mu_{q(b)} q(\mu_b^{-1}) \sigma_b \{[\iota_a] \cdot q(f)\} \sim_q [\iota_a] \diamond q(f). \end{aligned}$$

[†]And for simplicity we shall use the same symbol for the new functor.

In other words ι remains to be the natural transformation between the identity functor and q, in this new context. In a similar way we can prove that μ and σ remain to be natural tranformations in $\mathcal{C}^*$. In fact we shall prove a more general property:

Lemma 4.27. *Let Ψ be a natural isomorphism between functors q^n and q^m where $m, n \geq 1$, and such that morphisms $q(\Psi_c)\Psi^{-1}_{q(c)}\colon q^{m+1}(c) \to q^{m+1}(c)$ belong to the gauge group $\mathcal{G}_{q^m(c)}$ for every object $c \in \mathcal{C}$. Then Ψ is also a natural transfomation in $\mathcal{C}^*$.*

Proof. We have

$$\Psi_b q^n(f) = q^m(f)\Psi_a$$

for every $a, b \in \mathcal{C}$ and $f\colon a \to b$. Let us now consider a morphism in $\mathcal{C}^*$ between a and b represented by some $f\colon a \to q(b)$. We have

$$\begin{aligned} q^n(f)\diamond[\Psi_b] &= \mu_{q^m(b)}q(\iota_{q^m(b)}\Psi_b)q^n(f) = \mu_{q^m(b)}q(\iota_{q^m(b)})\{q(\Psi_b)\Psi^{-1}_{q(b)}\}q^m(f)\Psi_a \\ &= \mu_{q^m(b)}\sigma_{q^m(b)}\iota_{q^{m+1}(b)}\{q(\Psi_b)\Psi^{-1}_{q(b)}\}q^m(f)\Psi_a = \\ &= \mu_{q^m(b)}\sigma_{q^m(b)}q\{q(\Psi_b)\Psi^{-1}_{q(b)}\}q[q^m(f)]\iota_{q^n(a)}\Psi_a = \\ \{\mu_{q^m(b)}\sigma_{q^m(b)}q\{q(\Psi_b)\Psi^{-1}_{q(b)}\}\mu^{-1}_{q^m(b)}\}&\mu_{q^m(b)}q[q^m(f)]\iota_{q^n(a)}\Psi_a = u\big([\Psi_a]\diamond q^m(f)\big) \end{aligned}$$

where $u = \mu_{q^m(b)}\sigma_{q^m(b)}q\{q(\Psi_b)\Psi^{-1}_{q(b)}\}\mu^{-1}_{q^m(b)}$ is a natural transformation of q^{m+1} operator belonging to the gauge group of $q^m(b)$. □

It is interesting to observe that all gauge symmetries *vanish* in $\mathcal{C}^*$. To see this, let us consider an arbitrary $u \in \mathcal{G}_c$. It is a symmetry of $q(c)$. In $\mathcal{C}^*$ it will be represented as $\iota_{q(c)}u = q(u)\iota_{q(c)} \sim_q \iota_{q(c)}$ in other words, it is equivalent to the identity morphism of $q(c)$.

Let us now figure out how morphisms from $\mathcal{C}^*$ look like in $\mathcal{C}^{**}$. If $F\colon a \to b$ is a morphism in $\mathcal{C}^*$, it will be represented in $\mathcal{C}^{**}$ by the composition of F and ι_b in $\mathcal{C}^*$, that is, a $\mathcal{C}^*$-morphism from a to $q(b)$. If $f\colon a \to q(b)$ is a balanced $\mathcal{C}$-morphism representing F, then the mentioned $\mathcal{C}^*$-composition is represented by $\mu_b q(\iota_b)f \sim_q f$. This implies that $\mathcal{C}^*$ is faithfully represented in $\mathcal{C}^{**}$.

On the other hand, since $\mathcal{C}^*$ has no no-trivial gauge symmetries, a morphism $G\colon a \to b$ in $\mathcal{C}^{**}$ is reprented by a balanced morphism $F\colon a \to q(b)$

in $\mathcal{C}^*$ so that $[\Pi_b^-]F = F$. If $f\colon a \to q^2(b)$ is a balanced $\mathcal{C}$-morphism representing F, then this equality reads

$$\mu_{q(b)}q\big(\iota_{q(b)}\Pi_b^-\big)f = uf \qquad u \in \mathcal{G}_{qq(b)}$$

in $\mathcal{C}$. And transforming this we obtain

$$uf = \mu_{q(b)}q(\sigma_b)q^2(\iota_b)q(\Pi_b^-)f = \{\mu_{q(b)}q(\sigma_b)\mu_{q(b)}^{-1}\}q(\iota_b)\mu_b q(\Pi_b^-)f = \\ = \{\mu_{q(b)}q(\sigma_b)\mu_{q(b)}^{-1}\}\sigma_b\iota_{q(b)}\{\mu_b q(\Pi_b^-)f\}$$

which means that f is actually coming from a morphism $g\colon a \to q(b)$, and propagating back to $\mathcal{C}^*$ we conclude that F is coming from a morphism between a and b in $\mathcal{C}^*$. Summarizing everything, we conclude that categories $\mathcal{C}^*$ and $\mathcal{C}^{**}$ are in fact the same.

To conclude these categorical considerations, let us observe two natural scenarios for these constructions, in which we can further make some adjustments specific for the context of C*-algebras and quantum geometry. We shall meet both these scenarios in the next section.

The first scenario occurs when we start from a category say $\mathcal{W}$, but there is not 'enough room' to construct the functor q. In this case, we should first enlarge $\mathcal{W}$ to $\mathcal{C}$ by introducing new objects and morphisms between them, then construct $\mathcal{C}^*$, and finally restrict ourselves to the full subcategory $\mathcal{W}^*$ in $\mathcal{C}^*$ generated by $\mathcal{W}$.

The second interesting scenario is when morphisms from a to b in $\mathcal{C}^*$ provide 'too much' degrees of freedom. In this case we can consider some natural restrictions on balanced maps $f\colon a \to q(b)$ representing these morphisms.

5 C*-Algebraic Realizations

5.1 Multiplier Algebras & Woronowicz Category

Sometimes, it is necessary to enlarge the initial category, by realizing it as a full subcategory of another appropriately constructed extended category $\mathcal{C}$, in order to accomodate for the functor q, and its associated transformations.

Such a situation is illustrated by our present example, in which we shall introduce a category of quantum spaces, containing as a full subcategory the classical category of locally-compact spaces and their continuous maps. Everything is viewed in a dual, algebraic way, so the morphisms between spaces, correspond to morphisms between the corresponding algebras, going in the opposite directions. We follow here the principal construction of [27].

The objects of our category will be of two kinds. The first kind are C*-algebras A equipped with the strict topology induced from their multiplier algebra $M(A)$. These objects directly generalize a notion of 'space'. Non-unital algebras correspond to non-compact locally compact spaces, while unital algebras correspond to compact ones. Unital C*-algebras are characterized by $A = M(A)$ and in this case the strict topology and the norm topology coincide. In the non-unital case the two topologies on A will be very different.

The second kind of objects, are unital C*-algebras of the form $B = M(A)$ where A is non-unital, equipped with the strict topology. These objects generalize the Stone-Čech compactifications of non-compact locally-compact spaces. We can imagine them as carrying this extra structure, a compact 'space' encompassing a non-compact space, which is everywhere dense in its own enclosure.

It is worth mentioning here a fundamental property of the strict topology, that it can recover the initial algebra A inside of $M(A)$. Indeed, the elements of A are characterized as those elements $a \in M(A)$ for which the maps $b \mapsto ba$, or equivalently $b \mapsto ab$, are continous: the domain $M(A)$ is equipped with the strict topology, and the codomain $M(A)$ is equipped with the uniform one.

These are the objects of our category. The morphisms are, by definition, strictly continuous unital *-homomorphisms. In our non-necessarily unital context, unitality of a morphism means the intertwining of (possibly approximate) units.

So the objects of the first kind form a full subcategory $\mathcal{D}$ of $\mathcal{C}$. The morphisms of $\mathcal{D}$ generalize the idea of a proper continuous map between topological spaces. Indeed if A and B are commutative, then $A = C_0(X)$ and $B = C_0(Y)$ and every $F\colon A \to B$ which is a morphism in $\mathcal{D}$, is a pull-back of a proper continuous transformation $\Phi\colon Y \to X$.

The functor q acts on objects of $\mathcal{C}$ as follows. For every C*-algebra A, we have $q(A) = M(A)$, and in addition we assume that the additional topology type is preserved. This means that if A is non-unital then $M(A)$ is equipped with its strict topology, thus being an object of the second kind. If A is unital, then $M(A) = A$ and equipped with the uniform topology (obviously, coinciding with the strict). And if $A = M(B)$ for a non-unital B, then $q(A) = A = M(B)$, with the same strict topology.

Since A is always strictly dense in its completion $M(A)$, every strictly continuous map $F\colon A \to B$ uniquelly extends to a strictly continuous $q(F)\colon q(A) \to q(B)$. This completes our definition of the functor q. By construction $q^2 = q$.

The morphisms $\iota_A\colon A \to q(A)$ are given simply by the canonical inclusions of A into $M(A)$: let us recall that $M(A)$ is the maximal unital extension of A, in which A is viewed as an essential ideal. It is worth observing that these morphisms are 'going out' of the category $\mathcal{D}$, since they mix objects of the first and the second kind.

The maps $\mu_A\colon q^2(A) \to q(A)$ are identity maps. The twist operators are trivial, too. Hence the equivalence relation between morphisms will be trivial also, as we dont have any interesting transformations acting on objects $q(A)$. The balanceness condition is trivial, because the canonical projectors are identity maps.

The extended category $\mathcal{C}^*$ has the same objects, but enlarged morphisms base. By construction, the new morphisms between A and B are all unital strictly continuous maps $\Phi\colon A \to M(B)$. This is equivalent, according to our second interpretation of morphisms, to considering unital strictly continuous maps $\Phi\colon M(A) \to M(B)$, the continous extensions of the initial ones, and the composition is simply the standard composition of homomorphisms.

The objects of $\mathcal{D}$ equipped with new morphisms form a full subcategory $\mathcal{D}^*$ of $\mathcal{C}^*$. It is precisely the category of quantum spaces/pseudospaces of [27] generalizing the classical locally compact spaces and their continuous maps.

5.2 Extended Symmetries via Infinite Matrices

In this mathematical universe, the objects are given by appropriate C*-algebras A. The morphisms between A and B are unital *-homomorphisms from $M(A)$ to $M(B)$ which are strictly continuous. In other words, we consider a Woronowicz category as $\mathcal{C}$, discussed in the previous subsection. Actually we are dealing with a certain subcategory of this category, as the algebras A will not be arbitrary.

The functor q is given by tensoring by the ideal of compact operators K, acting in a fixed infinite-dimensional separable Hilbert space $\mathcal{H}$. So,

$$q(A) = A \otimes K \tag{5.1}$$

and the product $\otimes$ is the standard C*-algebraic tensor product. For a given morphism $F\colon M(A) \to M(B)$ we define $q(F)\colon q(A) \to q(B)$ to be the natural extension $q(F) \leftrightarrow F \otimes \mathrm{id}_K$. The morphisms $\iota_A\colon M(A) \to M(A \otimes K) = q(A)$ are the natural inclusion maps.

The twist operators $\sigma_A\colon M(A \otimes K \otimes K) \to M(A \otimes K \otimes K)$ are induced by the standard flip-over operator acting on $K \otimes K$.

The equivalences $\mu_A\colon M(A \otimes K \otimes K) \to M(A \otimes K)$ between objects $q^2(A)$ and $q(A)$ are induced by a fixed isomorphism $\cup\colon K \otimes K \to K$.

Let us recall that all automorphisms of K are represented by similarity transformations of unitary operators acting in $\mathcal{H}$. The algebras $A \otimes K$ and $M(A \otimes K)$ have a beautiful geometrical interpretation, as compact (closure of finite sums of Dirac diads) operators in $\ell^2(A)$ and adjointable (and necessarily bounded) operators in the Hilbert module $\ell^2(A)$ respectively.

In order to complete the picture, we have to specify all 'gauge symmetries' acting on objects $q(A)$. There are actually two very natural choices. Let us observe that all the necessary symmetry transformations coming from the general considerations, are representable as unitary transformations of $q(A)$ commuting with $M(A) \hookrightarrow M(A \otimes K)$. So one natural possibility is to define $\mathcal{G}_A$ as all possible such unitary transformations. Another natural possibility is to further extend the allowed symmetries, by defining $\mathcal{G}_A$ to be the group of all inner automorphisms of $M(A \otimes K)$. In other words $\mathcal{G}_A$ consists of all $\Phi\colon M(q(A)) \to M(q(A))$ given by

$$\Phi(a) = UaU^* \qquad U^* = U^{-1} \qquad U \in M(A \otimes K).$$

And we shall also include the appropriate strong limits of such inner automorphisms given by unitaries, so that we have a nice closed group. All automorphisms (true inner, and limits of those) will be simply called inner automorphisms. So to establish some property of a general inner automorphism, we start by approximating it via similarity transformations by unitary elements of $M(A \otimes K)$ and then considering the limit.

Here we shall focus on this second 'more symmetric' world, and let us describe in detail the morphisms of the category $\mathcal{C}^*$.

We start by considering morphisms $F\colon M(A) \to M(B \otimes K)$. Such maps are interpretable as non-degenerate representations of A in the Hilbert module $\ell^2(B)$. The balanceness condition says that F is 'truly infinite' in the sense of being equivalent to the infinite orthogonal sum of its copies. This corresponds to the identification $\ell^2(A) \leftrightarrow \ell^2(A) \otimes \ell^2(\mathbb{C})$. The equivalence $F \sim_q G$ means simply that representations F and G are unitary equivalent.

Our symmetries are given by unitary tranformations, and hence they are completely 'transportable': If ρ_U is the automorphism induced by a unitary U of $\ell^2(A)$ then $F\rho_U = \rho_{F(U)}F$. In this case, as we have seen, the double balanceness is equivalent to simple balanceness.

So our category deals with infinite representations of C*-algebras in Hilbert modules. A very interesting phenomenon occurs here: On one hand, we essentially enlarged the morphisms base by allowing all such representations in addition to morphisms of the Woronowicz category. And on the other hand, the enlarged symmetry framework, will make identifications between some different morphisms of $\mathcal{C}$. As a consequence of this $\mathcal{C}$ will not be faithfully represented via ζ as a subcategory of $\mathcal{C}^*$. The first, more restricted choice of 'gauge symmetries' would preserve the morphisms of $\mathcal{C}$.

Let us also observe that inner automorphisms are able to 'pass through' arbitrary morphisms—since the morphisms of our category are C*-algebraic homomorphisms. In particular, every morphism $F\colon q(A) \to q(B)$ of $\mathcal{C}$ is conductible.

Remark 5.1. Our mathematical universe has much more degrees of freedom, thanks to the enlarged operational space: The algebra $A \otimes K$ can be understood as 'infinite matrices' with entries in A. This is a natural playground for K-theory, Hilbert modules and Morita equivalence. Indeed,

two C*-algebras A and B are Morita equivalent precisely when $A \times K$ and $B \times K$ are isomorphic. This is equivalent to saying that there exists a Hilbert A-B bimodule $\mathcal{H}_{AB}$. The K-groups, and the bivariant KK-theory are all naturally formulated in the framework of this mathematical universe. See [26, 18] for more details.

Let us describe the monoidal structure in our category. For given A and B their product is simply the C*-algebraic tensor product. The functor q is made compatible with this product, via the following transformations

$$q(A)\otimes q(B) = A\otimes K\otimes B\otimes K \leftrightarrow A\otimes B\otimes K\otimes K \xrightarrow{\mu_{A\otimes B}} A\otimes B\otimes K = q(A\otimes B)$$

Now for $\mathcal{C}^*$-morphisms $f\colon M(A) \to M(C\otimes K)$ and $g\colon M(B) \to M(D\otimes K)$ then we can define their tensor product $f \otimes g$ as

$$M(A\otimes B) \xrightarrow{f\otimes g} M(C\otimes K\otimes D\otimes K) \leftrightarrow$$
$$\leftrightarrow M(C\otimes D\otimes K\otimes K) \xrightarrow{\mu_{C\otimes D}} M(C\otimes D\otimes K)$$

where in order to complicate things we used the same notation for different products. We can explicitly see that this product is compatible with the introduced gauge transformations and balanceness, but it also follows from the general considerations of the previous section. By passing to the quotient space of balanced morphisms, we obtain the tensor product of morphisms as classes.

The most interesting C*-algebras to deal with in this mathematical universe, are those that can be made compatible with themselves, in the sense of the following definition.

Definition 5.2. A C*-algebra A is called *q-friendly* iff there exists an inner automorphism ϕ of $A \otimes K$ such that

$$\phi(a)b = b\phi(a) \tag{5.2}$$

for every $a, b \in A$. Here we have extended ϕ to the multiplier algebra $M(A\otimes K)$ and viewed $M(A)$ as imbedded in $M(A\otimes K)$. In other words, the

entire algebra A can be displaced by ϕ into something that commutes with A. Every such an automorphism will be called a *displacement* automorphism for A. If ϕ is given by a unitary $U \in M(A \otimes K)$ so that $\phi(a) = UaU^*$, then we shall also call every such a unitary, a displacement unitary for A.

Let us observe that if (5.2) holds for elements of A then it holds for elements of $M(A)$ as well. This follows from the fact that A is dense in $M(A)$ in the strict topology, so we can simply take limits. We now present a series of simple lemmas about q-friendly algebras.

Lemma 5.1. *If A is q-friendly and ϕ its displacement morphism, then $\mu_A q(\phi)\mu_A^{-1}$ is a displacement morphism, too. Moreover, composing ϕ on the right with any inner automorphism by a unitary $u \in M(A)$, and on the left with any inner automorphism by a unitary $U \in M(A \otimes K)$ that commutes with A, produces again a valid displacement morphism.*

Proof. Since ϕ is inner, then $\mu_A q(\phi)\mu_A^{-1}$ is inner, too. It does the displacement job for A, because μ_A operates essentially on K, without touching A. The second part of the statement is a matter of a straightforward verification. □

Remark 5.3. If ϕ is a displacement automorphism for A, and u an arbitrary unitary in $M(A)$ then $x \mapsto \phi(uxu^*) = \phi(u)\phi(x)\phi(u)^*$ can be viewed in both ways, as the composition of the u-inner automorphism of $M(A)$ with ϕ, or the composition of ϕ with $U = \phi(u)$-inner automorphism of $M(A \otimes K)$. Let us observe also that U commutes with all $M(A)$.

Lemma 5.2. *If A and B are q-friendly then so are $A \otimes B$ and $A \oplus B$.*

Proof. Let U and V be the displacement operators for A and B. Then $U \oplus V$ is the displacement unitary for $A \oplus B$. Similarly, the displacement unitary for $A \otimes B$ is given by realizing the tensor product of U and V in $M(A \otimes B \otimes K \otimes K)$ and identifying $K \otimes K$ with K. □

Lemma 5.3. *If A is q-friendly and $F\colon A \to B$ an epimorphism, then B is q-friendly.*

Proof. Applying $F \otimes \mathrm{id}$ on (5.2) we obtain

$$VF(a)V^*F(b) = F(b)VF(a)V^* \qquad V = (F \otimes \mathrm{id})(U)$$

and we see that V is the displacement operator for B. □

Lemma 5.4. *Every commutative A is q-friendly. As are all matrix algebras $M_n(\mathbb{C})$ and the algebra K.*

Proof. If A is commutative then we can take U to be the identity, as $M(A)$ is the center of $M(A \otimes K)$. If $A = M_n(\mathbb{C})$ is the matrix algebra, we can use the standard twist acting in $A \otimes A$, and realize an identification $M_n(K) \leftrightarrow K$ to construct the displacement U. Finally, if $A = K$ then we can again use the standard twist in the first two factors in $M(A \otimes K \otimes K)$ and compose with $\mu_{A \otimes K}$. □

Lemma 5.5. *If $A \otimes K$ is q-friendly then so is A. Moreover, if A is q-friendly and B Morita equivalent to A, then B is q-friendly.*

Proof. Assuming that $A \otimes K$ is q-friendly, there exists a displacement unitary in $M(A \otimes K \otimes K)$. By composing with μ_A we obtain a displacement unitary for A. If A and B are Morita equivalent and A is q-friendly, then $A \otimes K$ is q-friendly, so is its isomorphic version $B \otimes K$, and so is B. □

Let us now consider a scenario in which direct limits of q-friendly algebras produce a q-friendly algebra. This in turn will imply that all approximately finite C*-algebras are q-friendly. By definition AF algebras are those that are viewable as direct limits of finite-dimensional C*-algebras. On the other hand, every finite-dimensional C*-algebra is viewable as a direct sum of matrix algebras. And, in accordance to the above results, is q-friendly with displacements generated by linear-algebraic twists. We shall start by considering an inductive limit sequence

$$A_0 \to A_1 \to A_2 \to \cdots \to A_n \to A_{n+1} \to \cdots \qquad \cdots \to A = \varinjlim A_n$$

of C*-algebras. By tensorising this by K we obtain another inductive limit sequence

$$\cdots \to A_n \otimes K \to A_{n+1} \otimes K \to \cdots \to A \otimes K = \varinjlim A_n \otimes K$$

and if the sequence is continuous in the strict topology, then there are naturally associated sequences of multiplier algebras

$$\cdots \to M(A_n) \to M(A_{n+1}) \to \cdots \to m(A) = \varinjlim M(A_n) \hookrightarrow M(A)$$
$$\cdots \to M(A_n \otimes K) \to M(A_{n+1} \otimes K) \to \cdots \to m(A \otimes K) \hookrightarrow M(A \otimes K)$$

Let us now assume that all A_n are q-friendly, with a displacement automorphism $\phi_n \colon A_n \otimes K \to A_n \otimes K$ given by a unitary operator U_n. In addition, let us assume that the displacements are compatible with the inductive limit sequence, so that

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_n \otimes K & \longrightarrow & A_{n+1} \otimes K & \longrightarrow \cdots \longrightarrow & A \otimes K \\ & & \phi_n \downarrow & & \downarrow \phi_{n+1} & & \downarrow \phi \\ \cdots & \longrightarrow & A_n \otimes K & \longrightarrow & A_{n+1} \otimes K & \longrightarrow \cdots \longrightarrow & A \otimes K \end{array}$$

with $\phi \colon A \otimes K \to A \otimes K$ being the induced automorphism. Such inductive limits are discussed in Appendix A. In particular, we conclude that ϕ is inner.

Lemma 5.6. *Under the above assumptions, the automorphism ϕ is a displacement automorphism for A and so A is q-friendly.*

Proof. Let $\mathcal{A}$ be the dense *-subalgebra of A generated by inclusions of A_n. Then we have

$$a\phi(b) = \lim a\phi_n(b) = \lim \phi_n(b)a = \phi(b)a$$

for every $a, b \in \mathcal{A}$. They always belong to the image of all A_n, for sufficiently large indexes n. By continuity (5.2) holds for all $a, b \in A$. □

Lemma 5.7. *Every AF C*-algebra is q-friendly.*

Proof. If A is AF then it can be viewed as a direct limit $A = \varinjlim A_n$ where each A_n is a finite-dimensional C*-algebra. In this case their displacement operators ϕ_n are very simple, and we can chose them in such a way that ϕ_n is compatible with the inductive sequence for $A \otimes K$. It follows that A is q-friendly. □

So the world of q-friendly creatures is quite large. However, not all C*-algebras are q-friendly. Extremely large algebras that are highly non-commutative will not be q-friendly: there will be no room for a proper displacement. For example, the algebra of all the bounded operators in a non-separable Hilbert space is not q-friendly.

Our category provides the necessary extension to construct the diagonal map, and thus formulate in a purely geometrical way the concepts of a quantum group and a quantum bundle. However, we have arguably introduced 'too much' freedom: If we consider classical spaces and morphisms between them, the new context provides a variety of extra morphisms between two non-trivial classical spaces. These extra morphisms really do not cause any harm, to the contrary, it is always helpful to play with more transformations.

On the other hand, if our goal is to include the classical geometry, as-it-is, within the new context, we have to provide an effective way to disregard these hidden transformations between classical spaces. This is achieved by the *halo algebra* construction.

Definition 5.4. For a given C*-algebra A, its *halo algebra* $H(A)$ is defined to be the minimal C*-algebra in $M(A \otimes K)$ containing $M(A)$ and invariant under all inner automorphisms of $M(A \otimes K)$.

Proposition 5.8. *The C*-algebra A is commutative iff $H(A) = M(A)$.*

Proof. If A is commutative then so is $M(A)$, and it actually forms the center of $M(A \otimes K)$. Thus the inner automorphisms act trivially on $M(A)$ and the halo algebra is the same as $M(A)$. Conversely, if the halo is trivial, then it implies that the similarity tranformation by every unitary 2×2 matrix with coefficients in $M(A)$ leaves diagonal matrices with coefficients in $M(A)$ invariant. In particular if $x = x^*$ is arbitrary self-adjoint element of $M(A)$ and U the canonical x-angle 2×2 rotation matrix associated to x, then

$$U \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} U^* =$$
$$= \begin{pmatrix} \cos(x)a\cos(x) + \sin(x)a\sin(x) & \cos(x)a\sin(x) - \sin(x)a\cos(x) \\ \sin(x)a\cos(x) - \cos(x)a\sin(x) & \cos(x)a\cos(x) + \sin(x)a\sin(x) \end{pmatrix}$$

for every $a \in M(A)$. So the preservation of diagonal matrices means in this particular case that

$$\sin(x)a\cos(x) = \cos(x)a\sin(x) \quad \Leftrightarrow \quad \frac{\sin(x)}{\cos(x)}a = a\frac{\sin(x)}{\cos(x)}$$

and we conclude[‡] that a commutes with tangens of self-adjoint elements, so it commutes with all self-adjoints, and so $M(A)$ must be commutative. □

So the existence of a non-trivial halo is a purely quantum phenomenon. The halo itself, turns out to be compatible with the canonical projection morphisms.

Lemma 5.9. *For every C^*-algebra A the halo algebra $H(A)$ is invariant under the projection morphism Π_A^-.*

Proof. The map $\iota_{q(A)}$ realizes $M(A \otimes K)$ naturally embedded in $M(A \otimes K \otimes K)$ via the first 2 tensor factors. The halo algebra $H(A)$ is then realized as the minimal subalgebra containing $M(A)$ and invariant under unitary transformations of the form $U \otimes \mathrm{id}$. After applying μ_A the image of $H(A)$ will be transformed into a minimal subalgebra of $M(A \otimes K)$ containing $M(A)$ and some of its unitary transforms–corresponding to the image of the above mentioned unitaries. By the definition of the halo, this image will be included in the halo of A itself. □

We can now introduce a natural restriction for morphisms of $\mathcal{C}^*$: they should map the initial algebra into the halo algebra.

Definition 5.5. A morphism $f\colon A \to q(B)$ is called proper, if its image is contained within $H(B)$.

In accordance with the above Lemma, we can always compose a proper morphism with the canonical projector, and obtain a proper balanced morphism. And by definition of the halo, it is invariant under all gauge symmetries. So we can extend the definition of properness to the whole classes of $\sim_q$.

[‡]Assuming here that $\cos(x)$ is invertible.

Lemma 5.10. *If $g\colon B \to q(C)$ is a proper morphism then $\Phi(g)\colon q(B) \to q(C)$ maps $H(B)$ into $H(C)$.*

Proof. Let us consider $q(g)\colon M(B\otimes K) \to M(C\otimes K\otimes K)$. It acts on $M(B)$ as g, if we view $M(C\otimes K)$ embedded in $M(C\otimes K\otimes K)$ via the first 2 tensor factors. In particular, it will map $M(B)$ into the image of $H(C)$. So the unitary transfoms of $M(B)$ in $M(B\otimes K)$ will be mapped into appropriate unitary transforms of $H(C)$ in $M(C\otimes K\otimes K)$. And if we compose everything with μ_C the image of $H(C)$ will go into $H(C)$, and the unitary transforms in $M(C\otimes K\otimes K)$ will be mapped into the corresponding unitary transforms of $H(C)$ in $M(C\otimes K)$. However the halo is always invariant under unitary transforms. □

Proposition 5.11. *The composition of proper morphisms is a proper morphism.*

Proof. Let us consider $f\colon A \to q(B)$ and $g\colon B \to q(C)$. Let us recall that their composition in $\mathcal{C}^*$ is represented by $\mu_C q(g) f = \Phi(g) f$. The statement directly follows from the previous lemma. □

If we now restrict our considerations to proper morphisms, we obtain a new category, which is still rich enough to adopt the diagonal maps and fibered products, since they all by construction 'live' in the halo algebras. On the other hand, the proper morphisms when restricted to the classical spaces, give us exactly the classical continuous transformations. In this sense we can thus say that the modified category faithfully represents the old, classical world.

On the other hand, by having less morphisms we have less capacity to compare things, and in particular isomorphic objects could turn out to be non-isomorphic in this restricted framework. Indeed, the concept of isomorphism here is something between C*-algebraic isomorphism and Morita equivalence. Consider, for example, proper morphisms between an arbitrary A and a commutative B. Since the halo is trivial for B, all morphisms are necessarily from the Woronowicz category. At the dual level of spaces, it means that we can retain the interpretation of points, as characters. So if A does not admit characters/points, then this is precisely a situation when

there will be no morphisms at all between A and B. In particular $A = B \otimes K$ and B are never isomorphic.

5.3 Diagrammatic Quantum Groups

We shall outline here the very basic elements of diagrammatic quantum groups of [8] and explain how they can be viewed as a realization of our categorical framework.

If a C*-algebra A represents a quantum space, in the spirit of the classical Gelfand-Naimark theory, then we shall assume that the complete geometrical structure of that space is specified by defining how the space behaves 'collectively' within itself. Let us consider a family of C*-ideals

$$J_n \subset \underbrace{A \star \cdots \star A}_{n} = A^{\star n}.$$

On these n-fold product algebras, we can define the action of a natural semigroup, generated by the product morphisms $m\colon A \star A \to A$, and also the permutation groups S_n naturally act here. We can furthermore consider embedding morphisms between different products.

So, the entire category of natural numbers and maps between them ($n \leftrightarrow \{0, 1, \ldots, n-1\}$), is represented in this context, by combining embeddings and the product morphisms m. We shall assume that the ideals J_n are invariant, in a natural way, under the action of all these morphisms.

To put it in other words, there are n different diagonal morphisms from $A^{\star(n+1)}$ to $A^{\star n}$. We require that all of them map J_{n+1} into J_n. Natural embedding maps $A^{\star n} \rightarrowtail A^{\star(n+1)}$ transform J_n into J_{n+1}. And the permutations acting in $A^{\star n}$ must preserve J_n. It is sufficient to assume this property for elementary transpositions.

We can define the 'higher collectivity modes' of the quantum space, as represented by the factor-algebras

$$A_n = A^{\star n}/J_n.$$

Our category $\mathcal{C}$ of quantum spaces is given by C*-algebras equipped with this collectivity information. The morphisms are *-homomorphisms

$M(A) \to M(B)$, which are unital strictly continuous and extendible to $M(A_n) \to M(B_n)$, as maps possessing the same properties.

An action of the multiplicative category $\mathbb{N}$ is given by $(n, A_m) \rightsquigarrow A_{nm}$ and morphisms are propagated in the most natural way. The algebras A_m should be here viewed as equipped with an induced collectivity substructure $(A_m)_n = A_{mn}$.

Remark 5.6. We can also consider an 'additive' version, in which the monoidal product is the sum (and of course, we should include zero in $\mathbb{N}$. In this case the natural choice is $(A_m)_n = A_{n+m}$, and develop such a concept of space, in which collectiveness 'remembers' individuality (so that A_d here are always refering to their individual counterpart A).

A compact matrix quantum group structure is introduced via a unitary matrix $u \in \mathrm{M}_d(A)$ the entries of which generate A, and a coproduct map $\phi\colon A \to A_2$ compatible with the collectivity structure, so that the entire diagrammatic symmetry category of [8] can be represented. This map should satisfy the matrix multiplication property

$$\phi(u_{ij}) = \sum_{k=1}^{d} [u_{ik}]_1 [u_{kj}]_2$$

in other words $\phi\colon u \rightsquigarrow u_1 u_2$. In this definition, the algebra is assumed to be unital.

Let us now describe a refined formulation suitable for 'fluctuating counit' examples (such as quantum torus and plane). In these examples, one 'coordinate' is left to represent unspecified 'point' of a quantum space that plays the role of the neutral element. This coordinate bears index zero, and the diagrammatic morphisms do not 'see' it, they act linearly over the zero-indexed copy of A. All other indexes correspond to 'active' collectivity modes, on which diagrammatic morphisms can non-trivially act. In this scenario, the coproduct is acting in the following way:

$$\phi(u_{ij}) = \sum_{k,l=1}^{d} [u_{ik}]_1 [u^*_{lk}]_0 [u_{lj}]_2$$

or in its matrix form $\phi\colon u \rightsquigarrow u_1 u_0^* u_2$.

Remark 5.7. In spite of a changed viewpoint, we are effectively within the same category: we just make a shift $n \mapsto n+1$ to accommodate this 'undefined point' in the space, modify the representation of the diagrammatic diagonal map, and introduce a different formula for the coproduct.

Remark 5.8. Both categories admit the diagonal map, induced by the (projected) multiplication *-homomorphism $m\colon A_2 \to A$. Indeed, the compatibility of the ideals J_n implies $m(J_2) = \{0\}$. The categorical diagonal map, introduced in Section 3, is equivalent to these, already existing at the non-extended category level, morphisms.

Remark 5.9. The diagrammatic definition of a quantum group is applicable to any of our categories. If the basic quantum space structure is specified by the action of $\mathbb{N}$ as a gauge symmetries category, then the group structure correspond to extensions of this action, to the enhanced category of diagrammatic morphisms.

5.4 Binary Sequences Quantum Playground

Let us consider the Woronowicz category $\mathcal{C}$. For a given C*-algebra A we shall denote by A^{op} its *opposite* algebra, where the product is the initial product in the reverse order. Equipped with the same *-structure and norm A^{op} is the C*-algebra as well. The multiplier algebra construction is symmetric relative to the exchange between A and A^{op}. In other words,

$$M(A)^{\mathrm{op}} = M(A^{\mathrm{op}}) \tag{5.3}$$

in a natural way, where the elements of $M(A)$, if interpreted as right multipliers acting on A, give the algebra $M(A^{\mathrm{op}})$. The strict topology is the same for both algebras.

We also have

$$(A \otimes B)^{\mathrm{op}} = A^{\mathrm{op}} \otimes B^{\mathrm{op}} \tag{5.4}$$

and every morphism $F\colon M(A) \to M(B)$ can be also interpreted as between the opposite algebras. Obviously if A is commutative (and only then) the identity map on A will be isomorphism between A and A^{op}, so we can write

$A = A^{\mathrm{op}}$ in this case. In general, algebras A and A^{op} will share the same center:

$$Z(A) = Z(A^{\mathrm{op}}). \tag{5.5}$$

Our covariant functor $q\colon \mathcal{C} \to \mathcal{C}$ will be the limit of a natural action of the blueprint category $\mathcal{N}$ defined in the following way. A given natural number n is first represented by a canonical binary sequence b_n of the length 2^n, as explained in detail in Appendix D. Then the C*-algebra $[n] \circ A$ is defined as a tensor product of copies of A and A^{op}, over the center algebra $Z(A) = Z(A^{\mathrm{op}})$. For example,

$$\varnothing \circ A = A \qquad [\bullet] \circ A = A \otimes_* A^{\mathrm{op}} \qquad [2] \circ A = A \otimes_* A^{\mathrm{op}} \otimes_* A^{\mathrm{op}} \otimes_* A. \tag{5.6}$$

Given a unital *-homomorphism $F\colon M(A) \to M(B)$ and a binary-sequence map $\mu\colon [n] \to [m]$ we can construct a unital *-homomorphism $\mu \circ F\colon M([n] \circ A) \to M([m] \circ B)$ by combining the tensor products of maps F with the injection of b_n into b_m, as dictated by μ.

The natural transformations ι, μ and σ are defined as in the previous section, as naturally associated to the action of $\mathcal{N}$ on $\mathcal{C}$. The system of gauge transformations can be always defined in the minimal way, as we explained in the previous section. In our context, this means all the symmetries induced by natural transformations ω and τ. We can also add the symmetries inherent in the category $\mathcal{N}$, which gives us the infinite permutation group acting on every object $q(A)$.

If the algebra A is commutative (and only in this case) we have

$$q(A) = A$$

and all the symmetries are trivial. So if we restrict to the subcategory of classical spaces, we see that it is completely preserved in this quantum context. The new morphisms would only appear when extend our view to include quantum spaces.

As discussed in in detail in [12] in our category every quantum space possesses a natural diagonal map, and we can always construct fibered products for bundles.

6 Concluding Remarks

In creating a geometry theory, the development of intuitive images, and visual representations, is a fundamental and integral part. Quantum geometry offers different, mutually related but inequivalent, interpretations for primary concept of space. Our constructions can be intuitively viewed, as extensions of such a vision of geometry, in which the fabric of space, besides its basic 'frozen' form, can 'oscillate' and produce more elaborate configurations. These 'oscillations' of the fabric of space can also be viewed as manifestations of kind of collectivity structure on the space, information about collective phenomena, represented by 'pra-objects' of the space. A kind of internal music, transformed into geometrical structure and forms.

The idea of 'objective' sound capable of generating forms, and in its more general interpretation, geometrical oscillations and vibrations of space, including the phenomena of light, generating forms, is present in an array of philosophical and mystical thoughts, cosmovisions, as well as in artistic creations [19].

Perhaps something along the lines of our concluding picture, which is a spherical adoption of the work [21] of Peter Neubacker on Harmonics.

Classical geometry does to much of an abstraction. The space as such is left without any physical attributes. Nikola Tesla always emphasized [25] the fundamental importance of *Luminous Ether*, as an indispensable ingredient of Geometry.

Our extended categorical framework, allows for a natural definition of diagonal maps and fiber products. This is essential in constructing a truly geometrical category of quantum spaces, groups and bundles. Let us elaborate on this more. Much more details with proofs will be presented in [12].

As we already mentioned in Subsection 5.2 where we have constructed a Morita-invariant category, we need to put some additional restrictions on the C*-algebras forming our initial category $\mathcal{C}$. The restriction is that the algebra A could be 'made compatible' with itself. In formal terms, we ask that there exists a unitary displacement element $U \in M(A \otimes K)$ such that

$$aUbU^* = UbU^*a$$

for every $a, b \in M(A) \hookrightarrow M(A \otimes K)$. It can be shown that such an operator always exists, for a very large class of C*-algebras. And moreover, any two of such inner automorphisms can be obtained one from another, by composing with an inner automorphism generated by a unitary $W \in M(A \otimes K)$ commuting with $M(A)$. This ensures the existence and uniqueness of the diagonal morphism in our category. Indeed the morphism $d_A \colon A \otimes A \to A$ is specified by the formula

$$d_A(a \otimes b) = aUbU^*.$$

The diagonal morphism is an intrinsic entity that we can associate to every quantum space. The category thus admits a natural formulation of the concept of a quantum group, by using the diagrammatic approach of [7]. Furthermore, there is a natural way to introduce the fibered product of quantum spaces, and in particular to introduce a coherent quantum geometrical framework for a digrammatic formulation of Dirac operators, developed in [22].

A similar definition applies for the infinite reflections scenario, as described in Subsection 5.4. These reflections can be interpreted in the spirit of [20], and as a new phenomenon ocurring only with truly quantum spaces. As elaborated in detail in Heisenberg's autobiographical notes [15], and illustrated in diverse discussions questioning the reality of atoms, in trying to understand a new world and reality with its completely new phenomena and appearences, we would start by using old expressions, however assigning to them completely new intuitive and operational meanings.

Our considerations also promote, a vision of the space in which its all regions are unified so to speak, or equivalently where there exist no way to introduce in any coherent manner, the divisibility in terms of parts, and local regions. For example, the quantum torus, for any irrational unitary number z, exhibits such a property. The C*-algebra will be simple. Another example of simple C*-algebras, is given by Cuntz algebras. They are also quite group-like objects, but not at all includable in the standard formulation. As elaborated in [1] this universal interconnectedness built in the fabric of geometry, including the symmetry between heaven and earth, was a fundamental invariant of the work of Remedios Varo, a great Catalan and Mexican artist.

Such algebras, viewed as algebraic representatives of quantum spaces, provide us with a natural framework to consider subquantum theories [13] in which the notion of classical locality is replaced by a more elaborate concept of a quantum geometricity. These theories unify principles of complementarity and superposition with the classical determinism. The obstacle of Bell's inequalities is overcome, by a switch in the geometrical interpretation of the spacetime continuum. Another, alternative way to proceed, is to consider a non-Kolmogorovian probability models, and this requires another radical change in the concept of a physical reality: That of a physical object, normally seen as something more or less isolated from its context-environment. In both scenarios, we have to abandon the idea of separability of things. Be it the space-time, as the most primitive geometrical environment for physical phenomena, or the very notion of a physical object.

A On Multiplier Algebras & Automorphisms

We collected here the basic facts of multiplier algebras. Our exposition of multiplier algebras closely follows [26]. We refer to beautiful [23] regarding more in-depth discussion on C*-algebra automorphisms.

Let A be a C*-algebra, and consider the following A-valued scalar product

$$\langle a, b\rangle = a^*b \tag{A.1}$$

defined on A. As a set $M(A)$ consists of all maps $T\colon A \to A$ for which there exists a map $T^*\colon A \to A$ such that

$$\langle a, T(b)\rangle = \langle T^*(a), b\rangle \tag{A.2}$$

for each $a, b \in A$.

Let us first observe that for a given T, its T^* is uniquely defined by the above formula so we have a unary operator $T \mapsto T^*$. Moreover T^* belongs to $M(A)$ as well, and we have $T^{**} = T$.

For every $T \in M(A)$ the identity

$$T(bc) = T(b)c \tag{A.3}$$

holds for all $b, c \in A$. We also have

$$T(b + \lambda c) = T(b) + \lambda T(c) \tag{A.4}$$

where $\lambda \in \mathbb{C}$. Indeed

$$\langle a, T(bc)\rangle = \langle T^*(a), bc\rangle = \langle T^*(a), b\rangle c = \langle a, T(b)\rangle c = \langle a, T(b)c\rangle$$

as well as

$$\begin{aligned}\langle a, T(b + \lambda c)\rangle = \langle T^*(a), b + \lambda c\rangle &= \langle T^*(a), b\rangle + \lambda\langle T^*(a), c\rangle \\ &= \langle a, T(b)\rangle + \lambda\langle a, T(c)\rangle = \langle a, T(b) + \lambda T(c)\rangle\end{aligned}$$

and the stated properties follow from the regularity of $\langle\,\rangle$.

Now if $T, S \in M(A)$ then TS as well as $T + \lambda S$ belong to $M(A)$, and moreover we have

$$(TS)^* = S^*T^* \qquad (T + \lambda S)^* = T^* + \bar{\lambda}S^* \tag{A.5}$$

and thus $M(A)$ equipped with the introduced algebraic structure, is a unital *-algebra. Another elementary calculation

$$\langle a, TS(b)\rangle = \langle T^*(a), S(b)\rangle = \langle S^*T^*(a), b\rangle$$

as well as

$$\begin{aligned}\langle a, (T + \lambda S)(a)\rangle = \langle a, T(b)\rangle + \lambda\langle a, S(b)\rangle &= \langle T^*(a), b\rangle + \lambda\langle S^*(a), b\rangle \\ &= \langle (T^* + \bar{\lambda}S^*)(a), b\rangle\end{aligned}$$

proves these.

So we have constructed a unital *-algebra of endomorphisms of the right A-module A. It turns out that all such maps T are automatically closed. Indeed, if $(a_n)_{n\in\mathbb{N}}$ is a sequence of elements of A converging to a and if we assume that $T(a_n)$ converges to b, then

$$\lim\langle c, T(a_n)\rangle = \lim\langle T^*(c), a_n\rangle = \langle c, b\rangle = \langle T^*(c), a\rangle$$

and it follows that $T(a) = b$, in other words T is closed. By applying the closed graphic theorem, we conclude that T is bounded. The norm of T is defined as

$$\|T\| = \sup_{a \neq 0} \frac{\|T(a)\|}{\|a\|}. \tag{A.6}$$

As a consequence of this definition, we immediately conclude that

$$\|T + S\| \leq \|T\| + \|S\| \qquad \|\lambda T\| \leq |\lambda| \|T\| \qquad \|TS\| \leq \|T\| \|S\| \tag{A.7}$$

in other words, the algebra $M(A)$ naturally becomes a normed algebra. The norm and the *-structure are related in the best possible, C*-way

$$\|T^*T\| = \|T\|^2. \tag{A.8}$$

Indeed, we have

$$\|T(a)\|^2 = \|\langle T(a), T(a)\rangle\| = \|\langle a, T^*T(a)\rangle\| \leq \|a\| \|T^*T(a)\| \leq \|a\|^2 \|T^*T\|$$

for all $a \in A$ and $T \in M(A)$, as it directly follows from the definition of the scalar product. This implies

$$\|T\|^2 \leq \|T^*T\|$$

however on the other hand

$$\|T^*T\| \leq \|T^*\| \|T\|$$

and it follows that $\|T\| \leq \|T^*\|$. By changing the roles of T and T^* we conclude that $\|T^*\| \leq \|T\|$ and hence $\|T\| = \|T^*\|$. Finally we arrive to the C*-equation (A.8).

Let us now observe that the norm on $M(A)$ is complete, that is, $M(A)$ is a Banach algebra. If $(T_n)_{n \in \mathbb{N}}$ is a Cauchy sequence of elements of $M(A)$ then $(T_n^*(a))_{n \in \mathbb{N}}$ as well as $(T_n(b))_{n \in \mathbb{N}}$ are Cauchy sequences of elements of A for every $a, b \in A$ and thus they converge in A. If we denote their limits by $T^*(a)$ and $T(b)$ we see that

$$\langle T^*(a), b\rangle = \lim \langle T_n^*(a), b\rangle = \lim \langle a, T_n(b)\rangle = \langle a, T(b)\rangle$$

so that both T and T^* are mutually adjoint elements of $M(A)$. It is easy to verify that $\lim T_n = T$ in the uniform operator topology.

Summarizing our construction thus far, we have a unital C*-algebra $M(A)$. It contains, in a natural way, the algebra A as its C*-subalgebra. To each $a \in A$ we can associate a map $b \mapsto ab$. In such a way A is isometrically realized as a *-subalgebra of $M(A)$. We have

$$M(A) = A \quad \Leftrightarrow \quad A \text{ is unital.} \tag{A.9}$$

It is obvious that if $M(A)$ and A coincide, then A is unital. On the other hand if $1 \in A$ then $T(a) = T(1)a$ for every $a \in A$ and we have $T \leftrightarrow T(1)$.

Furthermore A is an essential ideal of $M(A)$. The fact that A is a right ideal follows from the right linearity of elements of $M(A)$, as the property can be interpreted as

$$Ta \leftrightarrow T(a). \tag{A.10}$$

This immediately implies that if the product between T and all elements of A is zero, then $T = 0$. On the other hand

$$aT \leftrightarrow [T^*(a^*)]^* \tag{A.11}$$

so A is indeed a bilateral ideal in $M(A)$, and in fact an essential one.

The algebra $M(A)$ can be characterized as the maximal C*-algebraic extension of A, containing A as an essential ideal. Indeed if B is some other extension with this property, we can faithfully associate to every $b \in B$ a right-module endomorphism of A by $a \mapsto ba$. This gives a *-algebra inclusion $B \hookrightarrow M(A)$. Every such a *-algebra inclusion is isometric.

And if we now consider the multiplier algebra $M(B)$ of such a B, it is naturally includable in $M(A)$. Every map $T\colon B \to B$ from $M(B)$ must preserve A. This is because $T(ba) = T(b)a$ and for $a \in A$ we see that $T(ba) \in A$. On the other hand, the elements ba generate A. The restriction $T \mapsto T|_A$ gives the homomorphism of $M(B)$ into $M(A)$. It is injective, because if $T|_A = 0$ then $T(ba) = 0 = T(b)a$ and since A is essential in B, it follows that $T(b) = 0$ for every $b \in B$. The mentioned map is also surjective, if in addition B is an ideal in $M(A)$–necessarily essential since it contains A. This is because every element $S \in M(A)$ naturally acts on B via the left multiplication.

The algebra $M(A)$ possesses a very important topology, called *strict* topology. It is defined as the minimal topology in which all maps of the form

$$M(A) \ni b \mapsto ab \in A \qquad M(A) \ni b \mapsto ba \in A$$

are continuous. This topology is weaker then the uniform topology. It is strictly weaker iff $A \neq M(A)$. The *-operator is always continuous relative to the strict topology.

The algebra A is dense in $M(A)$, in the strict topology. Indeed we can always multiply the strictly convergent sequences on left and right by fixed elements of $M(A)$, obtaining again the strictly convergent sequences, and limits will commute with these operations. If $(e_\lambda)_{\lambda\in\Delta}$ is an approximate unit for A in $M(A)$ then it strictly converges to $1 \in M(A)$.

On the other hand, if the strict topology is given, then A is completely characterized within $M(A)$ in topological terms: it consists precisely of those elements $a \in M(A)$ for which the map $b \mapsto ab$ is continuous as a transformation from $M(A)$ equipped with the strict topology, to $M(A)$ equipped with its uniform topology. Indeed, if $a \in M(A)$ possesses this noble property, then the sequence $(au_\lambda)_{\lambda\in\Delta}$ of elements of A uniformly converges to a which means that $a \in A$.

If X is a locally compact topological space, then $A = C_0(X)$ is a commutative C*-algebra. It will be unital iff X is compact. According to the classical Gelfand-Naimark theory, all commutative C*-algebras A are canonically of this form. Given A the space X can be recovered as the set of characters of A equipped with the *-weak topology of the dual space A^*.

The algebra $M(A)$ is then naturally interpretable as $B(X)$, the continuous bounded functions $f\colon X \to \mathbb{C}$. The spectrum of $M(A)$ is exactly the Stone-Čech compactification $\tilde{X}$ of X. In other words $M(A) = B(X) = C(\tilde{X})$ in a natural way.

This can be further generalized to include into considerations B-valued maps on X where B is an arbitrary C*-algebra. The continuous functions $C_0(X, B)$ on X with values in B are naturally

$$C_0(X, B) \leftrightarrow C_0(X) \otimes B. \tag{A.12}$$

It turns out that the multiplier algebra of the above C*-algebra is given by

$$M[C_0(X,B)] = B[X, M(B)] \tag{A.13}$$

where $B[X, M(B)]$ are continuous bounded functions $f\colon X \to M(B)$ and $M(B)$ is considered as equipped with its strict topology.

Let us now consider automorphisms of a given C*-algebra A and their relations with the multiplier algebra $M(A)$. Every automorphisms $\phi\colon A \to A$ can be uniquely extended to an automorphism $\phi\colon M(A) \to M(A)$.

We shall consider a strong topology for morphisms: a sequence $(\phi_n)_{n\in\mathbb{N}}$ of automorphisms of A converges strongly to an automorphism ϕ iff

$$\lim \phi_n(a) = \phi(a)$$

for every $a \in A$, and the convergence is considered in the uniform topology of A.

Every unitary element $U \in M(A)$ gives rise to an automorphism $a \mapsto UaU^*$ of A. We shall call all these automorphisms inner automorphisms, and we shall also extend the terminology to cover the automorphisms that are strong limits of such inner automorphisms. They will be all called inner automorphisms.

Inner automorphisms form a closed invariant subgroup of all automorphisms. Let $\phi = \lim \phi_n$ with $\phi_n(a) = U_n a U_n^*$. Then for every automorphism ψ of A we have

$$\psi\phi\psi^{-1}(a) = \lim \psi(U_n)a\psi(U_n^*)$$

and therefore $\psi\phi\psi^{-1}$ is inner too.

If $\lim \phi_n = \phi$ in the strong topology sense described above, then we can also say that the convergence occurs at the level of the strict topology of $M(A)$. Indeed for an arbitrary $b \in M(A)$ we have

$$\lim \phi_n(b)\phi(a) = \phi(b)\phi(a) = \phi(ba) \qquad \phi(ab) = \phi(a)\phi(b) = \lim \phi(a)\phi_n(b).$$

To check the convergence in the strong topology of A, it is sufficient to verify it on elements of a dense *-subalgebra $\mathcal{A}$. If $\lim \phi_n(a) = \phi(a)$ for

every $a \in \mathcal{A}$ then for a given $\epsilon > 0$ and $b \in \mathcal{A}$ we can chose an a from $\mathcal{A}$ such that $\|b - a\| \leq \epsilon/4$ and for a sufficiently large n we will have

$$\|\phi(b) - \phi_n(b)\| \leq \|\phi(b) - \phi(a)\| + \|\phi(a) - \phi_n(a)\| + \|\phi_n(a) - \phi_n(b)\| \\ \leq \|b - a\| + \frac{\epsilon}{2} + \|a - b\| \leq \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon.$$

If $(\phi_n)_{n\in\mathbb{N}}$ are automorphisms strongly converging to a map $\phi\colon A \to A$ then ϕ is necessarily a *-monomorphism. We have

$$\|\phi(ab) - \phi(a)\phi(b)\| \leq \|\phi(ab) - \phi_n(ab)\| + \|\phi_n(a)\phi_n(b) - \phi_n(a)\phi(b)\| + \\ + \|\phi_n(a)\phi(b) - \phi(a)\phi(b)\| \leq \delta + \delta\|a\| + \delta\|b\| = \epsilon.$$

In other words ϕ is multiplicative and *-preserving because

$$\phi(a)^* = \lim \phi_n(a)^* = \lim \phi_n(a^*) = \phi(a^*).$$

By construction, ϕ is also isometric. However, this still does not imply that ϕ is an automorphism. This is why we have to put a specific condition for ϕ to be bijective.

It is easy to see that if $\lim \phi_n = \phi$ and $\lim \psi_n = \psi$ then $\lim \phi_n\psi_n = \phi\psi$. We have

$$\|\phi\psi(a) - \phi_n\psi_n(a)\| \leq \|\phi\psi(a) - \phi_n\psi(a)\| + \|\phi_n\psi(a) - \phi_n\psi_n(a)\| \\ \leq \|\phi\psi(a) - \phi_n\psi(a)\| + \|\psi(a) - \psi_n(a)\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for sufficiently large n. It follows that the inner automorphisms form a group.

The relevance of inner automorphisms is that they behave nicely with respect to the inductive limits of C*-algebras. Let us consider such a situation.

We shall assume that $A = \varinjlim A_n$ is an inductive limit C*-algebra such that all inclusion maps $A_n \rightsquigarrow A_{n+1}$ are continuous in the strict topology. So that they induce inclusion maps $M(A_n) \rightsquigarrow M(A_{n+1})$. We can form a direct limit $m(A) = \varinjlim M(A_n)$ and this will be a C*-subalgebra of $M(A)$, in a

natural way. A given sequence $(T_n)_{n\in\mathbb{N}}$ representing an element $T \in m(A)$ can be viewed as a multiplier $T\colon A \to A$ corresponding to

$$T(a) \leftrightarrow (T_n(a_n))_{n\in\mathbb{N}} \qquad a \leftrightarrow (a_n)_{n\in\mathbb{N}}.$$

If each A_n is equipped with an automorphism $\phi_n\colon A_n \to A_n$ such that the diagrams

$$\begin{array}{ccc} A_n & \longrightarrow & A_{n+1} \\ {\scriptstyle\phi_n}\downarrow & & \downarrow{\scriptstyle\phi_{n+1}} \\ A_n & \longrightarrow & A_{n+1} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} M(A_n) & \longrightarrow & M(A_{n+1}) \\ {\scriptstyle\phi_n}\downarrow & & \downarrow{\scriptstyle\phi_{n+1}} \\ M(A_n) & \longrightarrow & M(A_{n+1}) \end{array} \tag{A.14}$$

are commutative. Then there exists a unique $\phi\colon A \to A$ which is the automorphism of A induced by the system $(\phi_n)_{n\in\mathbb{N}}$. If ϕ_n are induced by unitaries $U_n \in M(A_n)$ then we can trivially extend these unitaries to unitaries $U_n \in M(A)$, by say

$$U_n \leftrightarrow [1 - p_n] \oplus U_n$$

where p_n is the projection in $M(A)$ defining the embedded $M(A_n)$. If ϕ_n is now the corresponding extended inner automorphism, then it is easy to see that $\lim \phi_n = \phi$ in other words, the direct limit automorphism is inner, too.

B On Commutative Banach Algebras

Lemma B.1. *Let A be a commutative unital Banach algebra, and $a \in A$ be a non-invertible element. If J is the closed ideal generated by A then $J \neq A$.*

Proof. If to the contrary $J = A$ then $1 \in A$ can be arbitrarily approximated by elements of the form $b = ar = ra$. For some $r \in A$ such an element would be invertible, which would imply the invertibility of $a \in A$. Indeed, if b is invertible then $a^{-1} = b^{-1}r = rb^{-1}$. □

Lemma B.2. *Let A be a commutative unital Banach algebra and J a non-trivial ideal in A. Then the closure of J is a non-trivial ideal, too. In particular, every maximal non-trivial ideal in A is closed.*

Proof. We have to discard the possibility that J is everywhere dense in A. Indeed, if we assume the opposite, then the unit $1 \in A$ can be arbitrarily approximated by the elements of J. This implies that there exists an invertible element in J, which is possible only if $J = A$. □

Lemma B.3. *If J is a maximal (thus closed) ideal in a commutative Banach algebra A, then naturally*

$$A/J \leftrightarrow \mathbb{C}.$$

In other words J is of the codimension one.

Proof. Because of maximality of J, there are no non-zero non-invertible elements in the factor-algebra A/J. This is only possible if $A/J \leftrightarrow \mathbb{C}$. □

C Polar Decompositions

In this Appendix we have collected some classical results regarding the polar decomposition of closed operators, and related properties involving commuting with bounded operators. We follow the classical exposition presented in [2].

Let $\mathcal{H}$ be a Hilbert space and Q a densely defined operator in $\mathcal{H}$. Then the adjoint operator $Q^\dagger$ exists, and its graph is defined by

$$g[Q^\dagger] = \circlearrowleft\{g[Q]^\perp\} \tag{C.1}$$

where $g[\,]$ is the graph operator, everything is happening in the direct sum Hilbert space $\mathcal{H} \oplus \mathcal{H}$ and $\circlearrowleft$ is the counterclockwise rotation for $\pi/2$ in $\mathcal{H} \oplus \mathcal{H}$, in other words $\circlearrowleft\colon (f, g) \mapsto (-g, f)$.

By construction the adjoint operator is always closed. It is densely defined, iff the operator Q is closable. In this case we have

$$g[\bar{Q}] = \overline{g[Q]} \qquad \bar{Q}^\dagger = Q^\dagger \qquad (Q^\dagger)^\dagger = \bar{Q}. \tag{C.2}$$

So if Q is closed, then $Q^\dagger$ is densely defined and the above geometrical representation implies that for every $f, g \in \mathcal{H}$ there exist unique $x \in D(Q)$ and $y \in D(Q^\dagger)$ such that

$$f = -Qx + y \qquad g = x + Q^\dagger y. \tag{C.3}$$

The domain $D(Q)$ is complete relative to the new scalar product given by

$$\langle \varphi, \psi \rangle_Q = \langle \varphi, \psi \rangle + \langle Q\varphi, Q\psi \rangle. \tag{C.4}$$

This scalar product is induced by the linear isomorphism $D(Q) \leftrightarrow g[Q]$ given by $x \leftrightarrow (x, Qx)$.

The operator $Q^\dagger Q$ is self adjoint. Its domain

$$D(Q^\dagger Q) = \left\{ x \in D(Q) \mid Qx \in D(Q^\dagger) \right\} \tag{C.5}$$

is dense in the domain of $D(Q)$, equipped with the above scalar product $\langle \rangle_Q$. The *modulus* of Q is defined as the unique positive self-adjoint operator $|Q|$ satisfying

$$|Q|^2 = Q^\dagger Q.$$

We then have

$$D(|Q|) = D(Q) \qquad \langle |Q|x, |Q|x \rangle = \langle Qx, Qx \rangle \quad \forall x \in D(Q) \tag{C.6}$$

and hence there exists a unique partial isometry U transforming isometrically $\mathrm{im}(|Q|)$ into $\mathrm{im}(Q)$ and such that $\ker(U) = \mathrm{im}(|Q|)^\perp$. The polar decomposition of Q is

$$Q = U|Q| \tag{C.7}$$

and we also have $\ker(U^\dagger) = \mathrm{im}(Q)^\perp$. The maps $U^\dagger U$ and $UU^\dagger$ are orthogonal projections on closures of $\mathrm{im}(|Q|)$ and $\mathrm{im}(Q)$ respectively.

Lemma C.1. *Let Λ be a bounded operator in $\mathcal{H}$, commuting with U and all of the spectral projectors of $|Q|$. Then,*

$$\Lambda D(Q) \subseteq D(Q) \qquad \Lambda Q \subseteq Q\Lambda. \tag{C.8}$$

In other words the domain of Q is Λ-invariant and Q commutes with the restriction of Λ on this space.

The polar decomposition effectively allows us to reduce the study of Q to a study involving much simpler operators: the partial isometry U and spectral projectors of the modulus $|Q|$.

Proof. Let us consider an increasing sequence $(e_n)_{n\in\mathbb{N}}$ of spectral projectors of $|Q|$ corresponding to intervals $[0,n]$. It strongly converges to the identity transformation in $\mathcal{H}$ and their images are all contained in the domain of Q. In particular for a given $\psi \in D(Q)$ and $n \in \mathbb{N}$ we have $e_n\psi \in D(\Lambda)$ and

$$e_n\psi \rightsquigarrow \psi \qquad D(Q) \ni e_n\Lambda\psi = \Lambda e_n\psi \rightsquigarrow \Lambda\psi$$
$$Qe_n\psi = U|Q|e_n\psi \rightsquigarrow U|Q|\psi = Q\psi$$
$$Q\Lambda e_n\psi = U|Q|e_n\Lambda\psi = \Lambda\{U|Q|e_n\psi\} = \Lambda Qe_n\psi \rightsquigarrow \Lambda Q\psi.$$

In other words $\Lambda\psi \in D(Q)$ and in $Q\Lambda\psi = \Lambda Q\psi$. □

Lemma C.2. *If Λ is a unitary operator in $\mathcal{H}$ such that Q is Λ-invariant, then Λ commutes with the partial isometry U and all of the spectral projectors of $|Q|$.*

Proof. The invariance of Q means

$$\Lambda Q\Lambda^* = Q = \{\Lambda U\Lambda^*\}\{\Lambda|Q|\Lambda^*\} \tag{C.9}$$

which, together with the uniqueness of the polar decomposition of Q implies that both U and $|Q|$ are Λ-invariant. In other words, $\Lambda U = U\Lambda$ and Λ commutes with all spectral projectors of $|Q|$. □

Remark C.1. The invariance property of Q is equivalent to requiring that both Λ and $\Lambda*$ reduce in $D(Q)$ and commute with Q on $D(Q)$.

Proposition C.3. *In particular if $\mathcal{R}$ is a von Neumann algebra then the following conditions are equivalent:*

(i) The partial isometry U and spectral projectors of $|Q|$ belong to $\mathcal{R}$;

(ii) Every element $\Lambda \in \mathcal{R}'$ satisfies the commutation condition

$$\Lambda D(Q) \subseteq D(Q) \qquad \Lambda Q \subseteq Q\Lambda. \tag{C.10}$$

(ii) The operator Q is Λ-invariant for every unitary Λ in $\mathcal{R}'$.

Proof. Everything follows by summarizing the previous two lemmas, and remembering that every element in a C*-algebra can be written as a linear combination of no more than 2 unitaries and their inverses. □

D Operations on Binary Sequences

Here we collected some elementary properties of sequences, which underline the main construction of infinite reflections algebras.

Let G be a groupoid, understood as a set with a binary operation. There exist two natural products operating in the spaces of sequences $\triangleleft, \triangleright\colon G^{\times n} \times G^{\times m} \to G^{\times nm}$ related with lexicographic ordering of sequences

$$(g_1, \ldots, g_n) \triangleright (h_1, \ldots, h_m) = (g_1 h_1, \ldots, g_1 h_m, \ldots, g_n h_1, \ldots, g_n h_m)$$
$$(g_1, \ldots, g_n) \triangleleft (h_1, \ldots, h_m) = (g_1 h_1, \ldots, g_n h_1, \ldots, g_1 h_m, \ldots, g_n h_m)$$

both products are associative in the space $G^{\times}$ of all finite sequences.

If the product in G is associative, then both $\triangleright$ and $\triangleleft$ are associative operations, that is to say

$$x \triangleleft (y \triangleleft z) = (x \triangleleft y) \triangleleft z \qquad x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright z \qquad \forall x, y, z \in G^{\times}. \quad \text{(D.1)}$$

The operations $\triangleright$ and $\triangleleft$ are mutually symmetric, it is easy to see that $x \triangleleft y$ becomes $y \triangleright x$ for the opposite product in G. So if G is Abelian then

$$x \triangleleft y = y \triangleright x \qquad \forall x, y \in G^{\times}. \quad \text{(D.2)}$$

We shall focus here on a special case $G = \mathbb{Z}_2 = \{+, -\}$. So all the sequences will be binary sequences of two symbols $+$ and $-$.

For every $n \in \mathbb{N}$ let us inductively define a binary sequence β_n of order 2^n as follows:

$$\beta_1 = (+, -) \qquad \beta_{n+1} = \beta_n \triangleleft (+, -) = (+, -) \triangleright \beta_n. \quad \text{(D.3)}$$

That is to say, every member of the sequence is obtained by *joining* the previous member with its opposite sequence. In particular we have

$$\beta_2 = (+, -, -, +) \qquad \beta_3 = (+, -, -, +, -, +, +, -)$$
$$\beta_4 = (+, -, -, +, -, +, +, -, -, +, +, -, +, -, -, +)$$
$$\beta_5 = \begin{array}{l}(+, -, -, +, -, +, +, -, -, +, +, -, +, -, -, + \\ -, +, +, -, +, -, -, +, +, -, -, +, -, +, +, -).\end{array}$$

It follows immediately from our inductive definition, that β_n is obtained by taking the n-fold power of $(+,-)$ relative to either $\triangleleft$ or $\triangleright$. Therefore the sequences β_n do not distinguish between these two products. Let us denote by $\cdot$ the common restriction of $\triangleleft$ and $\triangleright$. We thus have

$$\beta_n \cdot \beta_m = \beta_{n+m}. \tag{D.4}$$

In fact, the coincidence of products $\triangleleft$ and $\triangleright$ effectively characterizes the above sequences β_n, as it is perhaps worth noticing independently of our main context.

Lemma D.1. *The only binary sequences ψ that satisfy*

$$(+,-) \triangleleft \psi = (+,-) \triangleright \psi \tag{D.5}$$

are either $\psi = \beta_n$ or their opposites $\psi = -\beta_n$ for some $n \in \mathbb{N}$.

Proof. Let us first discard sequences of *odd* length. If $\psi = (a,b,c)$ then the above equation implies

$$(a,b,c,-a,-b,-c) = (a,-a,b,-b,c,-c)$$

which is clearly impossible. If $\psi \leftrightarrow (L,x,K)$ with $L = (l_1,\ldots,l_m)$ and $K = (k_1,\ldots,k_m)$ then

$$(L,x,K,-L,-x,-K) \leftrightarrow \big((+,-) \triangleleft L, x, -x, (+,-) \triangleleft K\big)$$

and it follows that

$$(l_1,-l_1,\ldots,l_m,-l_m,x) = (l_1,\ldots,l_m,x,k_1,\ldots,k_m) = \\ = (x,-k_1,k_1,\ldots,-k_m,k_m)$$

in particular we see that the above sequence is an alternating sequence.

In particular L and K are both alternating sequences, however in this case the first and the third representation fail to be alternating.

So ψ must be of the form $\psi \leftrightarrow (L,K)$ with L and K some sequences of the same order m. The main assumption implies

$$(L,K,-L,-K) \leftrightarrow (+,-) \triangleright \psi = (+,-) \triangleleft \psi \leftrightarrow \big((+,-) \triangleleft L, (+,-) \triangleleft K\big)$$

and therefore $K = -L$ and hence

$$(+,-) \triangleright L = (+,-) \triangleleft L = \psi.$$

In other words we can 'half' ψ and its half satisfies the same equation as ψ. So we can iterate this procedure, to conclude that the length of ψ must be 2^n, and there should be an initial sequence of length two satisfying the same equation (D.5). However this is only possible for $(+,-)$ and $(-,+)$. In summary, ψ is either β_n or $-\beta_n$ for some $n \in \mathbb{N}$. □

Definition D.1. We can introduce a concept of map/morphism between two binary sequences. For $n \leq m$ a monotone set map $F\colon \{1,\ldots,n\} \to \{1,\ldots,m\}$ is called a morphism between $(l_1,\ldots,l_n)$ and $(k_1,\ldots,k_m)$ iff $k_{F(o)} = l_o$ for every $o = 1,\ldots,n$.

Let $\mathcal{C}$ be the category of all finite binary sequences with morphisms as defined above. For a sequence x we shall also use $x^- = -x$ to denote the opposite sequence. For completeness we also include the two sequences of length one $\beta_0 = (+)$ and $\beta_0^- = (-)$ so that we have

$$x \triangleleft (+) = x \triangleright (+) = x \qquad x \triangleleft (-) = x \triangleright (-) = x^-.$$

Clearly, if F is a morphism between sequences x and y, then the same map can be interpreted as a morphism between x^- and y^-.

With respect to any of the products $\triangleleft$, $\triangleright$ and $\times$, the category $\mathcal{C}$ is a monoidal category, the operations being extendible to morphisms in a natural way.

Let us consider a subcategory $\mathcal{G}$ of $\mathcal{C}$, consisting of all objects β_n and β_n^- where $n \geq 0$ and morphisms defined as follows. A morphism between β_n or β_n^- and β_{n+k} or β_{n+k}^- is based on any (appropriate) map of the form $j \mapsto 2^k j + \ell - 2^k$ where $\ell = 1,\ldots 2^k$. In other words, we embed $\{1,\ldots,2^n\}$ into $\{1,\ldots,2^{n+k}\}$ in such a way that distance between neighbouring points is 2^k. Since in the 2^k-element intervals dividing the sequence $1,\ldots,2^{n+k}$ there will always be the same amount of $+$ and $-$ symbols, we conclude that there exist exactly 2^{k-1} morphisms between objects of grade n and $n+k$ for $k \geq 1$. In particular, there exists only one morphism between objects of grade n and $n+1$.

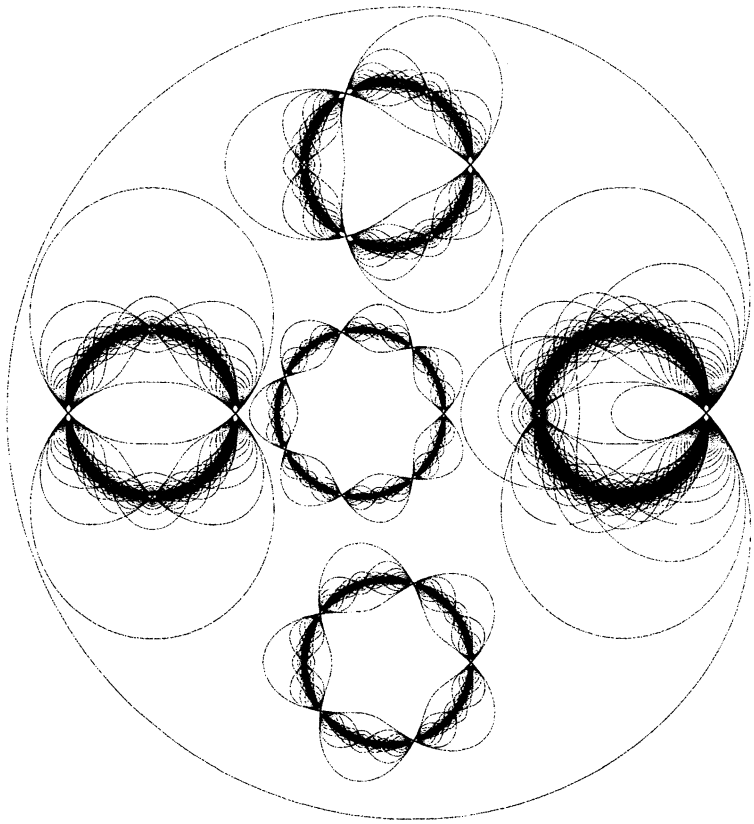
Lemma D.2. *If x is of grade n and y of grade $n+k$. Every morphism λ between x and y can be uniquely decomposed as*

$$\lambda = \lambda_1 \circ \cdots \circ \lambda_k \tag{D.6}$$

where $\lambda_i \colon z_i \to z_{i+1}$ are morphisms, $z_1 = x$ and $z_{k+1} = y$, and objects z_i are of grade $n+i-1$.

Proof. A simple combinatorial reasoning proves the statement. □

Since the number of intermediary steps $z_2, \ldots, z_k$ is $k-1$ and each step has two possibilities $z_i = \beta_i, \beta_i^-$ we have yet another explanation that the total number of morphisms is 2^{k-1}.



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QUANTUM RIEMANN SURFACES

Micho Đurđevich
Instituto de Matemáticas, UNAM
Area de la Investigación Científica
Circuito Exterior, Ciudad Universitaria
México DF, CP 04510, México
micho@matem.unam.mx
www.matem.unam.mx/~micho

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Abstract

We present a general theory of quantum Riemann surfaces, within the geometrical context of quantum frame principal bundles. All calculations are carried out in detail. They give us new insights into a geometrical concept of a surface. Besides the standard examples as the quantum 2-sphere coming from quantum Hopf fibration and quantum tori, our formalism naturally includes a variety of quantum objects related to interesting classical surfaces, as is the Klein tetric surface, and stochastic spheres with classical rotational symmetry, as well as the quantum hyperbolic and Euclidean planes. A quantum framework for all calculations involving the curvature and torsion tensors is developed.

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1 Introduction

The principal objective of this study is to develop, in a systematic way, a quantum-geometric framework for the study of objects that can be interpreted as quantum versions of Riemann surfaces.

We shall follow, at the quantum level, the same philosophy as introduced in the classical treatise [13]. The geometrical structures are viewed as certain bundles involving *frames*. So the principal playground is given by frame bundles, and in our quantum framework we shall built our theory

on the grounds of quantum frame bundles [8] which are a special subclass of general quantum principal bundles [3, 4, 8, 9]. There is an array of truly unique phenomena happening when the structure group of a given quantum principal bundle is classical.

The simplest non-trivial compact connected Lie group, is given by the circle, the rotational group SO(2). In this case there is still a plenty of room for studying purely quantum phenomena, for example, almost never in the quantum context the most appropriate calculus for a classical structure group will be the classical calculus. In this sense, we could say that a circle itself exhibits some 'hidden aspects' of its personality. On the other hand, being the circle group, its polynomial functions algebra $\mathcal{A}$ is generated by a single unitary element U and the coproduct $\phi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is given simply by $\phi(U) = U \otimes U$. So all the calculations can be carried out explicitly. An additional geometrical structure, besides the principal bundle structure, which justifies calling our objects 'quantum surfaces' will be given by a quantum frame system, consisting of two complementary 'derivations' acting on the bundle algebra. In the classical case, they correspond to the coordinate representation of the Levi-Civita connection.

The paper is organized in the following way. We shall start, for the sake of completeness, by overviewing the relevant aspects of the theory of quantum principal bundles, as developed and presented in detail in [3, 4, 8, 9]. A particular attention will be given to 'coordinate representation' of the Levi Civita connection, via the frame partial derivatives and their extension to the bundle algebra. In Section 3, we shall deal with certain bimodules, that can be interpreted as representing horizontal one-forms. The complex structure will be represented as an appropriate automorphism of this bimodule. We shall explain how the existence of a torsion-free covariant derivative implies non-trivial conditions on the algebraic structure of the mentioned bimodule. This analysis will naturally bring up a methodology of constructing interesting examples of quantum Riemann surfaces in which all covariant derivatives exhibit non-zero torsion.

We shall also consider gauge transformations and structural properties of the *-conjugation acting within the horizontal one-forms bimodule. A special case for surfaces, of a general reconstruction theory presented in [4], will be also discussed.

In Section 4, we shall deal with higher-order calculi. This is where phenomena related to torsion and curvature naturally take place. So we shall illustrate in complete detail, how the general expressions for all basic quantum geometrical objects look like in the context of our quantum Riemann surfaces. In Section 5 we have a more detailed look at the torsion tensor, and in particular prove the existence and uniqueness of the torsionless Levi-Civita connection.

Finally, in Section 6 some concluding observations are made. Two technical appendices complement the principal content of the paper. In the first one, we have collected and recalled some properties of derivations and the complex structure tensor, used in the main text. In the second one, we consider the Toepliz extension–as such a base for a nice example of quantum hyperbolic plane–and prove that the natural action of SO(2) on it, is not free.

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2 The Concept of a Quantum Surface

2.1 Preliminaries

We begin our journey by considering a general quantum principal bundle $P = (\mathcal{B}, \iota, F)$ over a quantum space $M \leftrightarrow \mathcal{V}$ and with the structure group $G = \mathrm{SO}(2)$. Intuitively, we can visualize a quantum family of oriented homogeneous circles of the same size. The algebra $\mathcal{A}$ of polynomial functions over G is generated by a single unitary element U, which is the canonical inclusion of the unitary circle $\mathrm{SO}(2) \hookrightarrow \mathbb{C}$ and also interpretable as its fundamental representation. The full set of irreducible representations is given by the powers U^n, where $n \in \mathbb{Z}$. Accordingly, the algebra $\mathcal{B}$ is naturally $\mathbb{Z}$-graded

$$\mathcal{B} = \bigoplus_{k\in\mathbb{Z}} \mathcal{B}_k \qquad \mathcal{B}_k = \left\{ b \in \mathcal{B} \,|\, F(b) = b \otimes U^k \right\} \tag{2.1}$$

and we have natural identifications

$$\mathcal{B}_0 = \mathcal{V} \qquad \mathcal{B}_n \otimes_{\mathcal{V}} \mathcal{B}_m \leftrightarrow \mathcal{B}_{n+m} \qquad \mathcal{B}_n^* = \mathcal{B}_{-n}. \tag{2.2}$$

The freeness property of the action F is equivalent to the existence of elements $b_1, \ldots, b_d \in \mathcal{B}_1$ and an invertible positive $\varrho \in \mathcal{V}$ so that

$$\sum_{\alpha=1}^{d} b_\alpha b_\alpha^* = 1 = \sum_{\alpha=1}^{d} \frac{1}{\varrho} b_\alpha^* b_\alpha \frac{1}{\varrho}. \tag{2.3}$$

Remark 1. The second equation in (2.3) reads $\sum_{\alpha=1}^{d} b_\alpha^* b_\alpha = \varrho^2$. Now, for any choice of the elements $b_\alpha \in \mathcal{B}_1$ the left hand side of this expression is a positive element of $\mathcal{V}$ and hence there exists a unique (positive) square root

$$\varrho = \sqrt{\textstyle\sum_{\alpha=1}^{d} b_\alpha^* b_\alpha}.$$

Therefore the really 'tricky' part of the above second freeness equation is the statement that such ρ is *strictly positive.* As we shall see in Appendix B there are interesting examples of (necessarily non-principal) quantum SO(2) bundles, in which the one equation holds, but the other can never be acomplished. Obviously, in the classical case the first equation implies the second (with $\varrho = 1$) so such a phenomenon of impossibility to satisfy both equations is *purely quantum.*

Remark 2. However, as it will be explained in the sequel of this essay, this essentially simplifies in a more general and truly geometric formulation, where bundles and groups are made compatible with the diagrammatic category describing quantum symmetries.

The elements figuring in (2.3) enable us to expand arbitrary elements of component modules $\mathcal{B}_k$ into the combinations of elements b_α or b_α^* with coefficients from the base algebra $\mathcal{V}$. Indeed, for $b \in \mathcal{B}_1$ we have

$$b = \sum_{\alpha=1}^{d} b_\alpha f_\alpha \qquad f_\alpha = b_\alpha^* b \in \mathcal{V} \tag{2.4}$$

and similarly

$$b\varrho = \sum_{\alpha=1}^{d} f'_\alpha b_\alpha \qquad f'_\alpha = b\frac{1}{\varrho}b^*_\alpha \in \mathcal{V} \tag{2.5}$$

A particular example of this are the rules of 'moving' the left products to right products, and viceversa, by elements of $\mathcal{V}$:

$$fb_\alpha = \sum_\beta b_\beta \rho_{\beta\alpha}(f) \qquad \rho_{\beta\alpha}(f) = b^*_\beta f b_\alpha \tag{2.6}$$

$$b_\alpha f = \sum_\beta \rho'_{\alpha\beta}(f) b_\beta \qquad \rho'_{\alpha\beta}(f) = b_\alpha f \varrho^{-2} b^*_\beta. \tag{2.7}$$

In the above formulae we have defined two homomorphisms $\rho/\rho' \colon \mathcal{V} \to \mathrm{M}_d(\mathcal{V})$. The first one is a *-homomorphism, while the second one satisfies $\rho'_{\alpha\beta}(f^*) = \rho'_{\beta\alpha}(\varrho^{-2} f \varrho^2)^*$ for every $f \in \mathcal{V}$.

2.2 Frame Partial Derivatives

Now from this $\mathbb{Z}$-graded algebra we proceed by gradually building the geometrical picture of a quantum Riemann surface. We shall conceptually follow a general theory of quantum frame bundles, however in order to make a self-contained exposition the basic steps will be repeated in this very particular and important context of quantum surfaces.

So our first important element of the structure is given by a pair of 'partial derivative' maps $\partial_1, \partial_2 \colon \mathcal{V} \to \mathcal{B}$ together with a system of four auxiliary linear maps $\epsilon_{11}, \epsilon_{12}, \epsilon_{21}, \epsilon_{22} \colon \mathcal{V} \to \mathcal{B}$ satisfying at first a *Leibniz rule* of the form

$$\begin{aligned} \partial_1(fg) &= f\partial_1(g) + \partial_1(f)\epsilon_{11}(g) + \partial_2(f)\epsilon_{21}(g) \\ \partial_2(fg) &= f\partial_2(g) + \partial_1(f)\epsilon_{12}(g) + \partial_2(f)\epsilon_{22}(g) \end{aligned} \tag{2.8}$$

for every $f, g \in \mathcal{V}$. Or simply

$$\vec{\partial}(fg) = \vec{\partial}(f)\epsilon(g) + f\vec{\partial}(g) \tag{2.9}$$

in a natural symbolic matrix representation, where

$$\vec{\partial} = (\partial_1, \partial_2) \qquad \epsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix}. \tag{2.10}$$

Secondly, we require the following *transformation properties* for the partials

$$\begin{aligned} F\partial_1(f) &= \partial_1(f) \otimes cos - \partial_2(f) \otimes sin \\ F\partial_2(f) &= \partial_1(f) \otimes sin + \partial_2(f) \otimes cos \end{aligned} \tag{2.11}$$

or more concisely in a symbolic matrix form

$$F\vec{\partial}(f) = \vec{\partial}(f) \begin{pmatrix} cos & sin \\ -sin & cos \end{pmatrix} \tag{2.12}$$

where the symbolic matrix product is taken over the corresponding tensor product.

Third, we shall assume that there exists a finite family of elements $f_\alpha = f_\alpha^* \in \mathcal{V}$ where $\alpha \in \{1, \ldots, d\}$ such that the following *coordinate completeness* property holds

$$\begin{aligned} \sum_\alpha \partial_1^*(f_\alpha)\partial_1(f_\alpha) &= 1 = \sum_\alpha \partial_2^*(f_\alpha)\partial_2(f_\alpha) \\ \sum_\alpha \partial_1^*(f_\alpha)\partial_2(f_\alpha) &= 0 = \sum_\alpha \partial_2^*(f_\alpha)\partial_1(f_\alpha). \end{aligned} \tag{2.13}$$

Fourth, we shall postulate the *minimality* condition, assuming that the algebra $\mathcal{V}$ is generated by elements $f_1, \ldots, f_d$.

So, in summary, we have these four conditions: Leibniz rule, Transformation property, Coordinate completeness and Minimality. A number of important explanatory remarks are in order.

An intuitive geometrical idea behind all this, is that ∂_1, ∂_2 represent partial derivatives associated to *oriented orthonormal frames* and functions f_α ensure an isometric 'embedding' of our quantum surface into a quantum d-dimensional Euclidean space. So the notion of the 'metric' is implicitly contained in combination of coordinate functions f_α and partial derivatives $\partial_{1,2}$, as explained in detail in the general quantum frame bundles theory. Furthermore just as in the general theory, the $2 \times d$ matrix

$$B = \begin{pmatrix} b_{11} & b_{12} \\ \vdots & \vdots \\ b_{d1} & b_{d2} \end{pmatrix} \tag{2.14}$$

with entries $b_{\alpha i} = \partial_i(f_\alpha)$ generate a canonical classifying map, from the surface to the quantum classifying space of the circle, and in particular relations (2.13) ensure the freeness of the action F of SO(2) on P.

These two conditions are actually not independent, the coordinate completeness implies that

$$\sum_\alpha \partial_i^*(f_\alpha)\partial_j(f_\alpha g) - \sum_\alpha \partial_i^*(f_\alpha) f_\alpha \partial_j(g) = \epsilon_{ij}(g) \tag{2.15}$$

so the map ϵ is completely determined by partial derivatives. The above expression in particular gives us the transformation properties of objects $\epsilon_{ij}(g)$: it is easy to see that

$$F\epsilon(g) = \begin{pmatrix} cos & -sin \\ sin & cos \end{pmatrix} \epsilon(g) \begin{pmatrix} cos & sin \\ -sin & cos \end{pmatrix} \tag{2.16}$$

for every $g \in \mathcal{V}$.

The minimality condition allows us to express the entire algebra $\mathcal{V}$ in terms of objects $f_1, \ldots, f_d$. Without such a condition, we would have much of an amiguity to define partial derivatives beyond the algebra generated by coordinate functions, and we could always focus on the subalgebra in $\mathcal{B}$ generated by f_α as well as $\partial_1(f_\alpha)$ and $\partial_2(f_\alpha)$ which would provide a minimal frame bundle structure. Geometrically, it means that the coordinate functions indeed realize an 'embedding' of M into the generic d-dimensional quantum Euclidean space, as we have already mentioned.

The coordinate completeness condition (2.13) could be generated from a more general *strict positivity* property

$$\begin{pmatrix} \partial_1^*(f_1) & \ldots & \partial_1^*(f_d) \\ \partial_2^*(f_1) & \ldots & \partial_2^*(f_d) \end{pmatrix} \begin{pmatrix} \partial_1(f_1) & \partial_2(f_1) \\ \vdots & \vdots \\ \partial_1(f_d) & \partial_2(f_d) \end{pmatrix} = T \tag{2.17}$$

where $T \in \mathrm{M}_2(\mathcal{B})$ is a strictly positive matrix possessing the same equivariance property

$$F(T) = \begin{pmatrix} cos & sin \\ -sin & cos \end{pmatrix} T \begin{pmatrix} cos & -sin \\ sin & cos \end{pmatrix} \tag{2.18}$$

as the values of ϵ, as it follows from the transformation properties of partial derivatives ∂_1 and ∂_2.

Remark 3. The above equation defines in fact an important C*-subalgebra $\mathcal{F}$ in $\mathrm{M}_2(\mathcal{B})$, and hence ϵ is in fact a homomorphism between $\mathcal{V}$ and $\mathcal{F}$.

The covariance condition defining $\mathcal{F}$ can be equivalently described as follows. Diagonalizing the fundamental representation of the circle

$$\frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}cos & -sin\\ sin & cos\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}^{*}=\begin{pmatrix}U & 0\\ 0 & U^{*}\end{pmatrix}$$

we see that equation (2.18) means that

$$F(\tau_{kl})=\tau_{kl}\otimes U^{2(k-l)}\qquad\qquad \begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}T\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}^{*} \tag{2.19}$$

in other words $\tau_{11},\tau_{22}\in\mathcal{V}$ while $\tau_{12}\in\mathcal{B}_{-2}$ and $\tau_{21}\in\mathcal{B}_2$.

So in our hypothetic more general case it is possible to 'rescale' partial derivatives into

$$\begin{pmatrix}\partial_1(f_1) & \partial_2(f_1)\\ \vdots & \vdots\\ \partial_1(f_d) & \partial_2(f_d)\end{pmatrix}\rightsquigarrow\begin{pmatrix}\partial_1(f_1) & \partial_2(f_1)\\ \vdots & \vdots\\ \partial_1(f_d) & \partial_2(f_d)\end{pmatrix}\frac{1}{\sqrt{T}} \tag{2.20}$$

and hence (2.13) holds for such renormalized maps. The ϵ is rescaled as

$$\epsilon\rightsquigarrow\sqrt{T}\epsilon\frac{1}{\sqrt{T}}. \tag{2.21}$$

Remark 4. In particular, we see that it is always possible to *enlarge* a given coordinate system $\{f_1,\ldots,f_d\}$ by arbitrarily chosen additional hermitian elements $\{f_{d+1},\ldots,f_{d+m}\}$ from $\mathcal{V}$.

The same coordinate completeness property ensures per se the *freeness* of the action of SO(2) on $\mathcal{B}$, so we necessarily have a principal bundle structure, and associated to it a *-morphism $\varrho\colon\mathcal{V}\to\mathrm{M}_d(\mathcal{V})$ defined by

$$\varrho_{\alpha\beta}(f)=\partial_1(f_\alpha)f\partial_1^{*}(f_\beta)+\partial_2(f_\alpha)f\partial_2^{*}(f_\beta) \tag{2.22}$$

with its support, an orthogonal projector

$$\Pi_{\alpha\beta}=\varrho_{\alpha\beta}(1) \tag{2.23}$$

in the algebra $\mathrm{M}_d(\mathcal{V})$. Equivalently we can use a matrix representation

$$\varrho(f) = BfB^* \qquad \Pi = BB^*. \tag{2.24}$$

Let us now discuss in more detail the Leibniz rule for partial derivatives. We shall first describe an equivalent version of a partial Leibniz rule, where the first factor belongs to the span of coordinate functions.

Lemma 1. *The following properties are equivalent:*

(i) A restricted Leibniz rule holds

$$\begin{aligned} \partial_1(f_\alpha g) &= f_\alpha \partial_1(g) + \partial_1(f_\alpha)\epsilon_{11}(g) + \partial_2(f_\alpha)\epsilon_{21}(g) \\ \partial_2(f_\alpha g) &= f_\alpha \partial_2(g) + \partial_1(f_\alpha)\epsilon_{12}(g) + \partial_2(f_\alpha)\epsilon_{22}(g) \end{aligned} \tag{2.25}$$

for each $g \in \mathcal{V}$ and $\alpha \in \{1, \ldots, d\}$.

(ii) We have

$$\begin{aligned} \partial_1(f_\alpha g) - f_\alpha \partial_1(g) &= \sum\nolimits_\beta \Pi_{\alpha\beta}\{\partial_1(f_\beta g) - f_\beta \partial_1(g)\} \\ \partial_2(f_\alpha g) - f_\alpha \partial_2(g) &= \sum\nolimits_\beta \Pi_{\alpha\beta}\{\partial_2(f_\beta g) - f_\beta \partial_2(g)\} \end{aligned} \tag{2.26}$$

in other words, a projection invariance in d-fold sum $\mathcal{B} \otimes \mathbb{C}^d$. Or

$$\varpi(g) = \Pi\varpi(g) \tag{2.27}$$

in a symbolic matrix representation where

$$\varpi = \begin{pmatrix} [\partial_1, f_1] & [\partial_2, f_1] \\ \vdots & \vdots \\ [\partial_1, f_d] & [\partial_2, f_d] \end{pmatrix} \tag{2.28}$$

and f_α interpreted as left multiplication operators.

Proof. The equivalence becomes evident, after substituting the formula (2.15) for ϵ in (2.25). In matrix representation we can simply write

$$\epsilon = B^*\varpi \tag{2.29}$$

and the restricted Leibniz rule reads

$$\varpi(g) = B\epsilon(g) \tag{2.30}$$

which, taking into account connection between B and Π, makes everything transparent. □

The most interesting situations are those when the partial derivatives obey the full Leibniz rule. This is not always the case, however it relates to the multiplicativity of ϵ, and it turns out that whenever the partial derivatives are compatible, in our Leibniz rule sense, with the quadratic combinations of coordinate functions, then the multiplicativity of ϵ and the full Leibniz rule hold.

Proposition 2. *The following properties are equivalent:*

(*i*) *The map* $\epsilon\colon \mathcal{V} \to \mathrm{M}_d(\mathcal{B})$ *is multiplicative.*

(*ii*) *We have*

$$\varpi(fg) = \varpi(f)B^*\varpi(g) \tag{2.31}$$

for every $f, g \in \mathcal{V}$.

(*iii*) *We have*

$$\begin{gathered}\varpi\big[f_{\alpha_1}\cdots f_{\alpha_m}\big] = \varpi(f_{\alpha_1})B^*\cdots B^*\varpi(f_{\alpha_m})\\ \varpi(f_\alpha) = BB^*\varpi(f_\alpha)\end{gathered} \tag{2.32}$$

for every index α *and multiindex* $(\alpha_1, \ldots, \alpha_m)$.

(*iv*) *We have*

$$\partial_j(\mu g) = \mu\partial_j(g) + \sum_{i=1}^{2} \partial_i(\mu)\epsilon_{ij}(g) \tag{2.33}$$

for every μ *quadratically expressible in terms of coordinate functions.*

(*v*) *The global Leibniz rule holds.*

Proof. It is clear that (2.31) expresses the multiplicativity of ϵ, because of (2.27) and (2.29). Taking into account that the coordinate functions

generate $\mathcal{V}$, the property (2.31) reduces to a bunch (2.32) of equations. Let us now prove that (iv) implies (iii). If (iv) holds then

$$\partial_j(f_\alpha f_\beta g) = \sum_{i=1}^{2} \partial_i(f_\alpha f_\beta)\epsilon_{ij}(g) + f_\alpha f_\beta \partial_j(g) = \sum_{ik=1}^{2} \partial_k(f_\alpha)\epsilon_{ki}(f_\beta)\epsilon_{ij}(g)$$

$$+ \sum_{i=1}^{2} f_\alpha \partial_i(f_\beta)\epsilon_{ij}(g) + f_\alpha f_\beta \partial_j(g) =$$

$$= \sum_{ik=1}^{2} \partial_k(f_\alpha)\epsilon_{ki}(f_\beta)\epsilon_{ij}(g) + f_\alpha \partial_j(f_\beta g) = \sum_{k=1}^{2} \partial_k(f_\alpha)\epsilon_{kj}(f_\beta g) + f_\alpha \partial_j(f_\beta g)$$

which, according to coordinate completeness property, implies that

$$\epsilon_{kj}(f_\beta g) = \sum_{i=1}^{2} \epsilon_{ki}(f_\beta)\epsilon_{ij}(g) \tag{2.34}$$

and from this, and the minimality condition for coordinate functions, it follows that ϵ is multiplicative. Let us denote by $\mathcal{F}$ a linear space of all elements $f \in \mathcal{V}$ for which the partials ∂_1 and ∂_2 satisfy the Leibniz rule for all $g \in \mathcal{V}$. It contains all coordinate functions and scalars. If we assume that ϵ is multiplicative, then considering arbitrary $f, h \in \mathcal{F}$ we can write

$$\overrightarrow{\partial}(fhg) = f\overrightarrow{\partial}(hg) + \overrightarrow{\partial}(f)\epsilon(hg) = f\overrightarrow{\partial}(h)\epsilon(g) + fh\overrightarrow{\partial}(g) + \overrightarrow{\partial}(f)\epsilon(h)\epsilon(g)$$

$$= \overrightarrow{\partial}(fh)\epsilon(g) + fh\overrightarrow{\partial}(g)$$

and therefore $\mathcal{F}$ is a subalgebra of $\mathcal{V}$. Since it contains the generators f_α we conclude that $\mathcal{F} = \mathcal{V}$ in other words the global Leibniz rule holds. □

Remark 5. Therefore in the case of multiplicative ϵ, equivalently the partials ∂_1 and ∂_2 satisfying the matrix Leibniz rule, the whole frame structure is determined by entities $\partial_1(f_\alpha)$ and $\partial_2(f_\alpha)$, which are entries of the matrix B as well as $\partial_1(f_\alpha f_\beta)$ and $\partial_2(f_\alpha f_\beta)$. Equivalently, because of the Leibniz rule, it is the matrix B together with the array $\epsilon(f_\alpha)$ completely defining the frame structure on the surface.

3 Horizontal One Forms

3.1 Coordinate Forms

From now on, we shall implement a very particular symmetry condition for the homomorphism ϵ. At first, we shall assume that the maps ϵ_{ij} take values in $\mathcal{V}$, so that we have a unital homomorphism $\epsilon\colon \mathcal{V} \to \mathrm{M}_2(\mathcal{V})$. In addition, we shall assume that

$$\epsilon(f) = \begin{pmatrix} c(f) & -s(f) \\ s(f) & c(f) \end{pmatrix} \tag{3.1}$$

where $c, s\colon \mathcal{V} \to \mathcal{V}$ are linear maps satisfying

$$\begin{aligned} c(fg) &= c(f)c(g) - s(f)s(g) \qquad & c(1) &= 1 \\ s(fg) &= s(f)c(g) + c(f)s(g) \qquad & s(1) &= 0. \end{aligned} \tag{3.2}$$

As it will soon become transparent, this condition is essential in developing a deeper geometrical structure, where we can introduce a splitting of the first-order horizontal forms, into the holomorphic and antiholomorphic parts. This will be possible iff

$$\epsilon(f) \begin{pmatrix} cos & -sin \\ sin & cos \end{pmatrix} = \begin{pmatrix} cos & -sin \\ sin & cos \end{pmatrix} \epsilon(f)$$

for all $f \in \mathcal{V}$, that is the matrices $\epsilon(f)$ should commute with the rotations. This leads to the above more restricted form

$$\epsilon_{11}(f) = \epsilon_{22}(f) = c(f) \qquad \epsilon_{12}(f) = -\epsilon_{21}(f) = s(f)$$

for the homomorphism ϵ. In terms of these functions the Leibniz rules for partial derivatives read

$$\begin{aligned} \partial_1(fg) &= f\partial_1(g) + \partial_1(f)c(g) + \partial_2(f)s(g) \\ \partial_2(fg) &= f\partial_2(g) - \partial_1(f)s(g) + \partial_2(f)c(g). \end{aligned} \tag{3.3}$$

We proceed to construct an invariant geometrical entity, the module $\mathcal{H}$ of *first order horizontal forms.* As a left $\mathcal{B}$-module it is free, and generated

by two coordinate forms θ_1 and θ_2. As far as the right $\mathcal{B}$-module structure is concerned, let us define it at the moment only for $f \in \mathcal{V}$ by postulating

$$\begin{aligned} \theta_1 f &= c(f)\theta_1 - s(f)\theta_2 \\ \theta_2 f &= c(f)\theta_2 + s(f)\theta_1 \end{aligned} \tag{3.4}$$

so up to now we have a 'modest' $\mathcal{B}$-$\mathcal{V}$ module.

Coordinate forms transform in accordance with the same rule as partial derivatives

$$\begin{aligned} \theta_1 &\rightsquigarrow \theta_1 \otimes cos - \theta_2 \otimes sin \\ \theta_2 &\rightsquigarrow \theta_1 \otimes sin + \theta_2 \otimes cos \end{aligned} \tag{3.5}$$

and this allows us to define the SO(2) symmetry $F_\wedge \colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{A}$ on all horizontal forms

$$F_\wedge(b\theta_{1,2}) = F(b)F_\wedge(\theta_{1,2}). \tag{3.6}$$

The constructed map indeed represents an action of G, in the sense of

$$(\mathrm{id} \otimes \epsilon)F_\wedge = \mathrm{id} \qquad (\mathrm{id} \otimes \phi)F_\wedge = (F_\wedge \otimes \mathrm{id})F_\wedge. \tag{3.7}$$

Let us now observe that by construction, the expression

$$d_M(f) = \partial_1(f)\theta_1 + \partial_2(f)\theta_2 \tag{3.8}$$

is $F_\wedge$-invariant, and the Leibniz rules for ∂_1 and ∂_2 combine with ϵ as the right multiplication by $\mathcal{V}$, into a standard derivation property

$$d_M(fg) = f d_M(g) + d_M(f)g \qquad \forall f, g \in \mathcal{V}. \tag{3.9}$$

And the orthogonality relations give us the following expressions of coordinate forms

$$\theta_1 = \sum_\alpha \partial_1^*(f_\alpha) d_M(f_\alpha) \qquad \theta_2 = \sum_\alpha \partial_2^*(f_\alpha) d_M(f_\alpha). \tag{3.10}$$

Let us now introduce new 'complex' coordinate forms and partial derivatives

$$\begin{aligned} \partial &= \frac{1}{2}(\partial_1 - i\partial_2) \qquad & \bar{\partial} &= \frac{1}{2}(\partial_1 + i\partial_2) \\ \theta &= \theta_1 + i\theta_2 & \bar{\theta} &= \theta_1 - i\theta_2 \end{aligned} \tag{3.11}$$

so that the Leibniz rules simplify into

$$\partial(fg) = f\partial(g) + \partial(f)\chi(g) \qquad \bar{\partial}(fg) = f\bar{\partial}(g) + \bar{\partial}(f)\bar{\chi}(g) \tag{3.12}$$

where $\chi, \bar{\chi}\colon \mathcal{V} \to \mathcal{V}$ are unital homomorphisms given by

$$\chi(f) = c(f) + is(f) \qquad \bar{\chi}(f) = c(f) - is(f). \tag{3.13}$$

In terms of these new 'complex' objects, the differential is given by

$$d_M(f) = \partial(f)\theta + \bar{\partial}(f)\bar{\theta} \tag{3.14}$$

and the right multiplication of coordinate forms becomes

$$\theta f = \chi(f)\theta \qquad \bar{\theta} f = \bar{\chi}(f)\bar{\theta}. \tag{3.15}$$

The maps ∂ and $\bar{\partial}$ are covariant in the sense of

$$F\partial(f) = \partial(f) \otimes U^* \qquad F\bar{\partial}(f) = \bar{\partial}(f) \otimes U \tag{3.16}$$

with $U = cos + isin$. In other words the derivation ∂ takes values from $\mathcal{B}_{-1}$ and $\bar{\partial}$ takes values from $\mathcal{B}_1$.

The coordinate completeness property can be expressed as

$$\begin{aligned} \sum_\alpha \partial^*(f_\alpha)\partial(f_\alpha) &= \frac{1}{2} = \sum_\alpha \bar{\partial}^*(f_\alpha)\bar{\partial}(f_\alpha) \\ \sum_\alpha \partial^*(f_\alpha)\bar{\partial}(f_\alpha) &= 0 = \sum_\alpha \bar{\partial}^*(f_\alpha^*)\partial(f_\alpha). \end{aligned} \tag{3.17}$$

The complex coordinate forms transform as

$$F_\wedge(\theta) = \theta \otimes U \qquad F_\wedge(\bar{\theta}) = \bar{\theta} \otimes U^*. \tag{3.18}$$

Let us denote by $\mathcal{M}$ the space of $F_\wedge$-invariant elements of $\mathcal{H}$. This is a $\mathcal{V}$-bimodule and we have

$$\mathcal{H} \hookleftarrow \mathcal{B} \otimes_{\mathcal{V}} \mathcal{M} \tag{3.19}$$

in a natural manner. We see that the differential d_M takes values from $\mathcal{M}$ and according to (3.17) we have

$$\sum_\alpha \partial^*(f_\alpha) d_M(f_\alpha) = \theta/2 \qquad \sum_\alpha \bar{\partial}^*(f_\alpha) d_M(f_\alpha) = \bar{\theta}/2. \tag{3.20}$$

In particular this gives the following decomposition for differentials of functions on the base:

$$d_M(f) = \sum_\alpha \{q_\alpha(f) + \bar{q}_\alpha(f)\} d_M(f_\alpha) \tag{3.21}$$

where 'projected partial derivative' operators acting in $\mathcal{V}$ are defined by

$$q_\alpha(f) = 2\partial(f)\partial(f_\alpha)^* \qquad \bar{q}_\alpha(f) = 2\bar{\partial}(f)\bar{\partial}(f_\alpha)^*. \tag{3.22}$$

Before going further a number of important observations are in order. The first two of the coordinate completeness conditions (3.17) are self-dual while the second two are mutually conjugate so there are really 3 independent conditions here.

And the first two equations ensure the *freeness* of the action of SO(2) on $\mathcal{B}$, so we indeed have a principal bundle structure, and associated to it a *-morphism $\varrho\colon \mathcal{V} \to \mathrm{M}_d(\mathcal{V})$ defined by

$$\varrho_{\alpha\beta}(f) = \partial_1(f_\alpha) f \partial_1(f_\beta)^* + \partial_2(f_\alpha) f \partial_2(f_\beta)^* = \varrho^+_{\alpha\beta}(f) + \varrho^-_{\alpha\beta}(f) \tag{3.23}$$

where maps ϱ^+ and $\varrho^-\colon \mathcal{V} \to \mathrm{M}_d(\mathcal{V})$ are given by

$$\varrho^+_{\alpha\beta}(f) = 2\partial(f_\alpha) f \partial(f_\beta)^* \qquad \varrho^-_{\alpha\beta}(f) = 2\bar{\partial}(f_\alpha) f \bar{\partial}(f_\beta)^*. \tag{3.24}$$

These maps are *-homomorphisms, as easily follows from the first line of (3.17). The second line tells us that these morphisms are *mutually orthogonal*. Their supports

$$\Pi^+ = \varrho^+(1) \qquad \Pi^- = \varrho^-(1) \tag{3.25}$$

are mutually orthogonal projectors in $\mathrm{M}_d(\mathcal{V})$ summing up to the support of ϱ, in other words we have

$$\begin{gathered} \Pi = \varrho(1) = \Pi^- + \Pi^+ \qquad \Pi^+_{\alpha\beta} = 2\partial(f_\alpha)\partial(f_\beta)^* \qquad \Pi^-_{\alpha\beta} = 2\bar{\partial}(f_\alpha)\bar{\partial}(f_\beta)^* \\ \Pi^{+*} = \Pi^+ = \Pi^+\Pi^+ \qquad \Pi^{-*} = \Pi^- = \Pi^-\Pi^- \qquad \Pi^+\Pi^- = \Pi^-\Pi^+ = 0. \end{gathered} \tag{3.26}$$

The derivatives q_α and $\bar{q}_\alpha$ satisfy the following matrix Leibniz identities

$$\begin{aligned} q_\beta(fg) &= fq_\beta(g) + \sum_\alpha q_\alpha(f)\varrho^+_{\alpha\beta}\chi(g) \\ \bar{q}_\beta(fg) &= f\bar{q}_\beta(g) + \sum_\alpha \bar{q}_\alpha(f)\varrho^-_{\alpha\beta}\bar{\chi}(g) \end{aligned} \tag{3.27}$$

with normalization conditions

$$q_\alpha(f_\beta) = \Pi^+_{\beta\alpha} \qquad \bar{q}_\alpha(f_\beta) = \Pi^-_{\beta\alpha}. \tag{3.28}$$

This in particular means that they are completely determined by their values on generators f_α.

3.2 Complex Structure Tensor

Our geometrical scenario is further elaborated by introducing the *complex structure tensor* $J\colon \mathcal{H} \to \mathcal{H}$ which is a left $\mathcal{B}$-module automorphism defined by

$$J(\theta) = i\theta \quad J(\bar{\theta}) = -i\bar{\theta} \qquad \Leftrightarrow \qquad J(\theta_1) = -\theta_2 \quad J(\theta_2) = \theta_1 \tag{3.29}$$

and by construction this map satisfies a covariance property

$$F_\wedge J = (J \otimes \mathrm{id})F_\wedge. \tag{3.30}$$

By definition $J^2 = -1$ and the complex structure J has two eigenvalues i and $-i$ corresponding to horizontal forms decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \qquad \mathcal{H}_+ = \mathcal{B}\theta \qquad \mathcal{H}_- = \mathcal{B}\bar{\theta} \tag{3.31}$$

to type $(1,0)$ and type $(0,1)$ horizontal forms respectively. The covariance property for J implies that it reduces in $\mathcal{M}$ and the following decompositions hold

$$\begin{aligned} \mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_- \qquad & \mathcal{H}_\pm \rightsquigarrow \mathcal{B} \otimes_\nu \mathcal{M}_\pm \\ \mathcal{M}_+ = \mathcal{B}_{-1}\theta \qquad & \mathcal{M}_- = \mathcal{B}_1\bar{\theta} \end{aligned} \tag{3.32}$$

which is in harmony with the definition of the differential d_M, as we can write

$$\begin{aligned} d_M(f) &= d^+_M(f) + d^-_M(f) \\ d^+_M(f) = \partial(f)\theta \qquad & d^-_M(f) = \bar{\partial}(f)\bar{\theta} \end{aligned} \tag{3.33}$$

that is, a sum of holomorphic and antiholomorphic parts.

The differentials d_M^- and d_M^+ are standard derivations valued in $\mathcal{M}^-$ and $\mathcal{M}^+$ respectively. The following commutation relations hold

$$\begin{aligned} d_M^+(f_\alpha)f &= \sum_\beta \varrho_{\alpha\beta}^+ \chi(f) d_M^+(f_\beta) \\ d_M^-(f_\alpha)f &= \sum_\beta \varrho_{\alpha\beta}^- \bar{\chi}(f) d_M^-(f_\beta) \end{aligned} \tag{3.34}$$

which in particular imply that

$$\begin{aligned} \sum_\alpha \Pi_{\alpha\beta}^+ d_M^+(f_\beta) &= d_M^+(f_\alpha) \qquad \sum_\alpha \Pi_{\alpha\beta}^- d_M^+(f_\beta) = 0 \\ \sum_\alpha \Pi_{\alpha\beta}^+ d_M^-(f_\beta) &= 0 \qquad \sum_\alpha \Pi_{\alpha\beta}^- d_M^-(f_\beta) = d_M^-(f_\alpha). \end{aligned} \tag{3.35}$$

As we have already emphasized, it is precisely this decomposition into holomorphic and antiholomorphic parts, that should be preserved as an invariant of the geometric structure. This implies the special form of ϵ, in terms of the 'trigonometric' morphisms $c, s \colon \mathcal{V} \to \mathcal{V}$.

3.3 Gauge Transformations

It is interesting to ask to what extent the coordinate forms θ and $\bar{\theta}$ are determined geometrically, within the calculus we have just described. In fact, as easily follows from our constructions, the only freedom we have left is to perform a simple *gauge transformation* by independent unitary elements $u, v \in \mathcal{V}$, so that

$$\theta \rightsquigarrow u\theta \qquad \bar{\theta} \rightsquigarrow v\bar{\theta}. \tag{3.36}$$

We would like to have d_M remain gauge-invariant, so this leads to the following reparametrizations of basic structural entities:

$$\begin{aligned} \partial(f) &\rightsquigarrow \partial(f)u^* \qquad \chi(f) \rightsquigarrow u\chi(f)u^* \\ \bar{\partial}(f) &\rightsquigarrow \bar{\partial}(f)v^* \qquad \bar{\chi}(f) \rightsquigarrow v\bar{\chi}(f)v^*. \end{aligned} \tag{3.37}$$

The unitarity property of u and v is a necessary condition in order to have the invariance of relations (3.17) of coordinate completeness under such gauge transformations. It is also important to observe the complete gauge invariance of the morphisms $\varrho^+\chi, \varrho^-\bar{\chi}\colon \mathcal{V} \to \mathrm{M}_d(\mathcal{V})$ as well as the 'projected' partial derivatives $q_\alpha\colon \mathcal{V} \to \mathcal{V}$ and $\bar{q}_\alpha\colon \mathcal{V} \to \mathcal{V}$ so that we have, in particular the following gauge-invariant expressions for holomorphic and antiholomorphic differentials

$$d_M^+(f) = \sum_\alpha q_\alpha(f) d_M^+(f_\alpha) \qquad d_M^-(f) = \sum_\alpha \bar{q}_\alpha(f) d_M^-(f_\alpha) \tag{3.38}$$

these operators being already implicitly present in (3.14) and (3.21).

Remark 6. Differentials d_M^+ and d_M^- can be actually *defined* by the above expressions. Their standard Leibniz identities logically follow from the respective matrix Leibniz rules in (3.27) coupled with the commutation relations (3.34) defining the right multiplication by elements of $\mathcal{V}$ of differentials of coordinate functions.

Consequently our full group of gauge transformations is the direct product $\mathcal{G} = \mathrm{U}(\mathcal{V}) \times \mathrm{U}(\mathcal{V})$. This group will be dealt with further, when we introduce a *-conjugation in the game, which will also allow us to promote the horizontal one-forms space into a nice $\mathcal{B}$-bimodule.

3.4 Bimodule & *-Structures

Now we shall consider *-structures and prove that the calculus given by $\mathcal{M}$ and d_M is indeed a *-calculus. It worth recalling that such a *-structure on $\mathcal{M}$ is unique, if it exists. And it is necessary given by assignements

$$\begin{aligned} \sum_\alpha \{b\partial(f_\alpha)^*\} d_M(f_\alpha) &\rightsquigarrow \sum_\alpha d_M(f_\alpha)\{\partial(f_\alpha) b^*\} \\ \sum_\alpha \{q\bar{\partial}(f_\alpha)^*\} d_M(f_\alpha) &\rightsquigarrow \sum_\alpha d_M(f_\alpha)\{\bar{\partial}(f_\alpha) q^*\} \end{aligned} \tag{3.39}$$

where $b \in \mathcal{B}_{-1}$ and $q \in \mathcal{B}_1$. The above expressions on the other hand, consistently define an antilinear map $*\colon \mathcal{M} \to \mathcal{M}$ which by construction

satisfies $(f\psi)^* = \psi^* f^*$ for every $f \in \mathcal{V}$ and $\psi \in \mathcal{M}$. We will have the *-calculus, iff such a map is involutive. This is equivalent to $(\psi f)^* = f^*\psi^*$. Yet another equivalent formulation is to say that $d_M(f^*) = d_M(f)^*$ for every $f \in \mathcal{V}$. However we are looking for a *-structure that mixes holomorphic and antiholomorphic forms $\mathcal{M}_+$ and $\mathcal{M}_+$, which is expressible as an additional condition

$$J* = *J \quad \Leftrightarrow \quad d_M^+(f^*) = d_M^-(f)^* \quad \forall f \in \mathcal{V}. \tag{3.40}$$

This implies a non-trivial additional relation between holomorphic and antiholomorphic objects.

Lemma 3. *A necessary and sufficient condition for the existence of such a natural *-structure is that*

$$\begin{aligned} \sum_\alpha \varrho^+_{\alpha\beta}\chi\big[\varrho^{-*}_{\alpha\gamma}\bar{\chi}^*(f)\big] &= f\Pi^+_{\gamma\beta} \\ \sum_\alpha \varrho^-_{\alpha\beta}\bar{\chi}\big[\varrho^{+*}_{\alpha\gamma}\chi^*(f)\big] &= f\Pi^-_{\gamma\beta} \end{aligned} \tag{3.41}$$

hold for every $f \in \mathcal{V}$. In particular

$$\sum_\alpha \varrho^+_{\alpha\beta}\chi\big[\Pi^-_{\alpha\gamma}\big] = \Pi^+_{\gamma\beta} \qquad \sum_\alpha \varrho^-_{\alpha\beta}\bar{\chi}\big[\Pi^+_{\alpha\gamma}\big] = \Pi^-_{\gamma\beta}. \tag{3.42}$$

Proof. This is a simple exercise in matrix calculus and describing idempotency conditions in coordinate ways. □

The *-structure we have just established acts on $\mathcal{M}$. Let us now analyze its extendibility from $\mathcal{M}$ to $\mathcal{H}$. This is intrinsically related to the existence of an appropriate $\mathcal{B}$-bimodule structure on $\mathcal{H}$. Indeed, the *-operator intertwines the left and right multiplications by elements from $\mathcal{B}$, so knowing the left multiplication we can reconstruct the right and vice versa. On the other hand, we would like to have holomorphic and antiholomorphic parts of $\mathcal{H}$ mixed by the extended *-structure. This implies that $\mathcal{H}_-$ and $\mathcal{H}_+$ are full $\mathcal{B}$-sumbodules of $\mathcal{H}$, and in particular, the morphisms χ and $\bar{\chi}$ are extendible from $\mathcal{V}$ to $\mathcal{B}$ so that we have

$$\theta b = \chi(b)\theta \qquad \bar{\theta} b = \bar{\chi}(b)\bar{\theta} \tag{3.43}$$

for every $b \in \mathcal{B}$. In other words, the above formula fixes the right $\mathcal{B}$-module structure on $\mathcal{H}_-$ and $\mathcal{H}_+$. This multiplication should be compatible with the action of $G = \mathrm{SO}(2)$ on $\mathcal{H}$ so that

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F} & \mathcal{B}\otimes\mathcal{A} \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi\otimes\mathrm{id}} \\ \mathcal{B} & \xrightarrow[F]{} & \mathcal{B}\otimes\mathcal{A} \end{array} \qquad \begin{array}{ccc} \mathcal{B} & \xrightarrow{F} & \mathcal{B}\otimes\mathcal{A} \\ {\scriptstyle\bar\chi}\downarrow & & \downarrow{\scriptstyle\bar\chi\otimes\mathrm{id}} \\ \mathcal{B} & \xrightarrow[F]{} & \mathcal{B}\otimes\mathcal{A} \end{array}$$

Equivalently, the extended morphisms preserve the spaces $\mathcal{B}_k$ where $k \in \mathbb{Z}$.

These conditions imply a strong interrelation between holomorphic and antiholomorphic objects, as we are now going to explain. By construction we have

$$\theta^* = \rho\bar\theta \qquad \bar\theta^* = \bar\rho\theta \tag{3.44}$$

where ρ and $\bar\rho$ are invertibles of $\mathcal{V}$ related by

$$\bar\rho = \chi(\frac{1}{\rho^*}) \qquad \rho = \bar\chi(\frac{1}{\bar\rho^*}). \tag{3.45}$$

In playing with the above equations we can take advantage of the polar decomposition of invertible elements of $\mathcal{V}$, and write

$$\begin{array}{ll} \rho = |\rho|\upsilon & \bar\rho = |\bar\rho|\bar\upsilon \\ |\rho| = \sqrt{\rho\rho^*} & |\bar\rho| = \sqrt{\bar\rho\bar\rho^*}. \end{array} \tag{3.46}$$

We see that the unitary parts υ and $\bar\upsilon$ can be interpreted as gauge transformations acting on antiholomorphic or holomorphic forms. So we can always assume that one of them is trivial, that is either ρ or $\bar\rho$ is (strictly–because of invertibility) positive. However, in general they will not be *both* positive at the same time, which is an interesting purely quantum phenomena. For this reason, we shall keep both ρ and $\bar\rho$ in the 'general' form and the theory will exhibit a nice explicit symmetry between holomorphic and antiholomorphic sectors. Now going 'full circle' with our *-operator we obtain

$$b\theta \rightsquigarrow \theta^*b^* = \rho\bar\theta b^* = \rho\bar\chi(b^*)\bar\theta \rightsquigarrow \bar\theta^*\bar\chi(b^*)^*\rho^* = \chi(\frac{1}{\rho^*})\theta\bar\chi(b^*)^*\rho^* =$$

$$= \chi\Big\{\frac{1}{\rho^*}\bar\chi^*(b)\rho^*\Big\}\theta$$

in other words

$$\bar{\chi}(b) = \frac{1}{\rho}\chi^{-1*}(b)\rho \qquad \chi(b) = \frac{1}{\bar{\rho}}\bar{\chi}^{-1*}(b)\bar{\rho} \tag{3.47}$$

so the pairs morphism-element (χ, ρ) and $(\bar{\chi}, \bar{\rho})$ completely determine each other. Furthermore the intertwining property for differentials d_M^+ and d_M^- reads

$$d_M^+(f)^* = \left[\partial(f)\theta\right]^* = \theta^*\partial(f)^* = \rho\bar{\chi}\{\partial(f)^*\}\bar{\theta} = \bar{\partial}(f^*)\bar{\theta} = d_M^-(f^*)$$

and therefore

$$\begin{aligned} \bar{\partial}(f) &= \rho\bar{\chi}\{\partial^*(f)\} = \chi^{-1*}\{\partial^*(f)\}\rho = \{[\chi^{-1}\partial]^*(f)\}\rho \\ \partial(f) &= \bar{\rho}\chi\{\bar{\partial}^*(f)\} = \bar{\chi}^{-1*}\{\bar{\partial}^*(f)\}\bar{\rho} = \{[\bar{\chi}^{-1}\bar{\partial}]^*(f)\}\bar{\rho}. \end{aligned} \tag{3.48}$$

The partial derivatives ∂ and $\bar{\partial}$ therefore determine each other. It should not come as a big surprise, that the relation between triplets (∂, χ, ρ) and $(\bar{\partial}, \bar{\chi}, \bar{\rho})$ is completely symmetrical.

Lemma 4. *The automorphism χ and derivative ∂ completely determine the modulus of ρ, as we have*

$$\frac{1}{|\rho|} = \sqrt{2\sum\nolimits_\alpha [\chi^{-1}\partial(f_\alpha)]\,[\chi^{-1}\partial(f_\alpha)]^*}. \tag{3.49}$$

In a similar way

$$\sqrt{2\sum\nolimits_\alpha [\bar{\chi}^{-1}\bar{\partial}(f_\alpha)]\,[\bar{\chi}^{-1}\bar{\partial}(f_\alpha)]^*} = \frac{1}{|\bar{\rho}|}. \tag{3.50}$$

Proof. The coordinate completeness for $\bar{\partial}$ together with third and fourth expressions in (3.48) gives

$$\frac{1}{2} = \sum_\alpha \bar{\partial}(f_\alpha)^*\bar{\partial}(f_\alpha) = \sum_\alpha \rho^*[\chi^{-1}\partial(f_\alpha)]\,[\chi^{-1}\partial(f_\alpha)]^*\rho$$

and the first formula for the modulus follows. The second one is obtained by transposing holomorphic and antiholomorphic objects. □

Let us return for a moment to our discussion of gauge transformations. Because of the *-structure, the holomorphic and antiholomorphic objects do not transform independently, the transformation in one sector fixes the transformation in another. Therefore the group $\mathcal{G}$ of gauge transformations is determined by the action on triplets (χ, ∂, ρ) or triplets $(\bar{\chi}, \bar{\partial}, \bar{\rho})$. As we already explained, the group can be viewed as the direct product of the unitary group of $\mathcal{V}$ by itself.

From the already analyzed transformational properties

$$\partial \rightsquigarrow \partial u^* \qquad \chi \rightsquigarrow u\chi u^* \qquad \theta \rightsquigarrow u\theta \qquad \bar{\theta} \rightsquigarrow v\bar{\theta} \qquad \bar{\chi} \rightsquigarrow v\bar{\chi}v^* \qquad \bar{\partial} \rightsquigarrow \bar{\partial}v^*$$

it follows easily that

$$\rho \rightsquigarrow \rho\bar{\chi}(u^*)v^* \qquad \bar{\rho} \rightsquigarrow \bar{\rho}\chi(v^*)u^*$$

and in particular we see that the action of $\mathcal{G}$ on ∂ completely determine the unitary u and v is then determined from the transformed ρ. Similarly, transformed $\bar{\rho}$ tells us what u is, assuming that we have already determined v from the transformed $\bar{\partial}$.

3.5 Reconstruction Problematics

To round up the present section, and before moving to higher-order calculus, we shall here depict explicitly how starting from ∂ and χ we obtain a consistent first-order horizontal *-calculus. So let us begin at first with an automorphism $\chi\colon \mathcal{B} \to \mathcal{B}$ which intertwines the circle action F. This is equivalent to saying that $\chi(\mathcal{B}_k) = \mathcal{B}_k$ for each $k \in \mathbb{Z}$, in particular χ reduces in $\mathcal{V} = \mathcal{B}_0$. And a right χ-derivation $\partial\colon \mathcal{V} \to \mathcal{B}$ with values in $\mathcal{B}_{-1}$ and satisfying

$$\sum_\alpha \partial(f_\alpha)^*\partial(f_\alpha) = \frac{1}{2} \qquad \sum_\alpha \partial'(f_\alpha)\partial(f_\alpha) = 0 \tag{3.51}$$

where $\partial' = \chi^{-1}\partial$. Let us introduce an invertible element $\rho \in \mathcal{V}$ such that

$$|\rho| = \Big[2\sum_\alpha \partial'(f_\alpha)\partial'^*(f_\alpha)\Big]^{-1/2} \tag{3.52}$$

so the element ρ is determined up to its argument. One natural choice would be to have this unitary part trivial, so that $\rho = |\rho|$, but any ρ would do fine.

Remark 7. In fact, we have to postulate, and this would be our last assumption here, the strict positiveness of $\sum_\alpha \partial'(f_\alpha)\partial'^*(f_\alpha)$ which is equivalent to the strict positiveness of $\sum_\alpha \partial(f_\alpha)\partial^*(f_\alpha)$.

Now we can define the antiholomorphic derivative $\bar\partial\colon \mathcal{V} \to \mathcal{B}$ and its automorphism $\bar\chi\colon \mathcal{B} \to \mathcal{B}$ by

$$\bar\partial(f) = \partial'^*(f)\rho \qquad \bar\chi(b) = \frac{1}{\rho}\chi^{-1*}(b)\rho. \tag{3.53}$$

As a matter of direct verification, we conclude that $\bar\partial$ is a right $\bar\chi$-derivation with values in $\mathcal{B}_1$. The following completeness relations follow directly from the corresponding relations for the holomorphic partial derivative:

$$\sum_\alpha \bar\partial^*(f_\alpha)\partial(f_\alpha) = 0 = \sum_\alpha \partial^*(f_\alpha)\bar\partial(f_\alpha) \qquad \sum_\alpha \bar\partial^*(f_\alpha)\bar\partial(f_\alpha) = \frac{1}{2}. \tag{3.54}$$

So we can proceed in a similar way as already described, and introduce symbols θ and $\bar\theta$ to represent coordinate forms, and define a covariant $\mathcal{B}$-*-bimodule $\mathcal{H}$ of first-order horizontal forms, as

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \qquad \mathcal{H}_+ = \mathcal{B}\theta \qquad \mathcal{H}_- = \mathcal{B}\bar\theta$$

with the right $\mathcal{B}$-bimodule structure specified by

$$\theta b = \chi(b)\theta \qquad \bar\theta b = \bar\chi(b)\bar\theta$$

and the *-structure specified by

$$\theta^* = \rho\bar\theta \qquad \bar\theta^* = \bar\rho\theta \qquad \bar\rho = \chi(\frac{1}{\rho^*}).$$

By construction the operator * intertwines $\mathcal{H}_+$ and $\mathcal{H}_-$. It is indeed a bimodule *-structure, as

$$(b\theta)^{**} = [\theta^* b^*]^* = (\rho\bar\chi(b^*)\bar\theta)^* = \bar\theta^*\bar\chi^*(b)\rho^* = \bar\rho\theta\bar\chi^*(b)\rho^* = \theta\chi^{-1}(b) = b\theta$$

in other words * is involutive. By construction, the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{F_\sim} & \mathcal{H}\otimes\mathcal{A} \\ {\scriptstyle *}\downarrow & & \downarrow{\scriptstyle *\otimes *} \\ \mathcal{H} & \xrightarrow[F_\sim]{} & \mathcal{H}\otimes\mathcal{A} \end{array} \tag{3.55}$$

is commutative, in other words the *-operation is covariant with respect to the circle action, and in particular the invariant $\mathcal{V}$-submodule $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$ is *-invariant. The calculus structure is given by already familiar expressions

$$d_M^+(f) = \partial(f)\theta \qquad d_M^-(f) = \bar\partial(f)\bar\theta$$

and we have

$$d_M^+(f)^* = (\partial(f)\theta)^* = \theta^*\partial(f)^* = \rho\bar\chi\{\partial(f)^*\}\bar\theta = \\ = \chi^{-1*}\{\partial(f)^*\}\rho\bar\theta = \bar\partial(f^*)\bar\theta = d_M^-(f^*).$$

Playing with the differentials and the bimodule relations, it is easy to establish the following properties

$$\begin{array}{ll} \mathcal{B}d_M^+(\mathcal{V}) = d_M^+(\mathcal{V})\mathcal{B} = \mathcal{H}_+ & \mathcal{V}d_M^+(\mathcal{V}) = d_M^+(\mathcal{V})\mathcal{V} = \mathcal{M}_+ \\ \mathcal{B}d_M^-(\mathcal{V}) = d_M^-(\mathcal{V})\mathcal{B} = \mathcal{H}_- & \mathcal{V}d_M^-(\mathcal{V}) = d_M^-(\mathcal{V})\mathcal{V} = \mathcal{M}_- \end{array} \tag{3.56}$$

with a mutual independence of holomorphic and antiholomorphic parts

$$\mathcal{B}d_M(\mathcal{V}) = d_M(\mathcal{V})\mathcal{B} = \mathcal{H} \qquad \mathcal{V}d_M(\mathcal{V}) = d_M(\mathcal{V})\mathcal{V} = \mathcal{M}. \tag{3.57}$$

The maps d_M^+ and d_M^- take values from the corresponding invariant parts $\mathcal{M}_+ \subset \mathcal{H}_+$ and $\mathcal{M}_- \subset \mathcal{H}_-$ and we see that $d_M = d_M^+ + d_M^-$ endowes $\mathcal{M}$ with the structure of a *-calculus over $\mathcal{V}$. In terminology of the general theory [4] we have constructed a horizontal first-order calculus for the bundle P.

It is instructive to explicitly check the validity of the gauge invariant

relations established in Lemma 3. We have

$$\sum_\alpha \varrho^+_{\alpha\beta}\chi\big[\varrho^-_{\alpha\gamma}\bar\chi^*(f)\big] = 4\sum_\alpha \partial(f_\alpha)\chi\Big[\bar\partial(f_\alpha)\bar\chi^*(f)\bar\partial^*(f_\gamma)\Big]\partial^*(f_\beta)$$

$$= 4\Big\{\sum_\alpha \partial(f_\alpha)\chi\big[\bar\partial(f_\alpha)\big]\Big\}\chi\bar\chi^*(f)\chi\big\{\bar\partial^*(f_\gamma)\big\}\partial^*(f_\beta) = 2\bar\rho\frac{1}{\bar\rho}f\bar\rho\chi\big\{\bar\partial^*(f_\gamma)\big\}\partial^*(f_\beta)$$

$$= 2f\bar\rho\chi\big\{\rho^*\chi^{-1}\partial(f_\gamma)\big\}\partial^*(f_\beta) = 2f\partial(f_\gamma)\partial^*(f_\beta) = f\Pi^+_{\gamma\beta}$$

and in the same way—exchanging holomorphic and antiholomorphic objects

$$\sum_\alpha \varrho^-_{\alpha\beta}\bar\chi\big[\varrho^+_{\alpha\gamma}\chi^*(f)\big] = 4\sum_\alpha \bar\partial(f_\alpha)\bar\chi\Big[\partial(f_\alpha)\chi^*(f)\partial^*(f_\gamma)\Big]\bar\partial^*(f_\beta)$$

$$= 4\Big\{\sum_\alpha \bar\partial(f_\alpha)\bar\chi\big[\partial(f_\alpha)\big]\Big\}\bar\chi\chi^*(f)\bar\chi\big\{\partial^*(f_\gamma)\big\}\bar\partial^*(f_\beta) = 2\rho\frac{1}{\rho}f\rho\bar\chi\big\{\partial^*(f_\gamma)\big\}\bar\partial^*(f_\beta)$$

$$= 2f\rho\bar\chi\big\{\bar\rho^*\bar\chi^{-1}\bar\partial(f_\gamma)\big\}\bar\partial^*(f_\beta) = 2f\bar\partial(f_\gamma)\bar\partial^*(f_\beta) = f\Pi^-_{\gamma\beta}$$

where we have used the composition formulae

$$\chi\bar\chi^*(b) = \frac{1}{\bar\rho}b\bar\rho \qquad \bar\chi\chi^*(b) = \frac{1}{\rho}b\rho$$

together with identities

$$\sum_\alpha \partial(f_\alpha)\chi\big[\bar\partial(f_\alpha)\big] = \frac{1}{2}\bar\rho \qquad \frac{1}{2}\rho = \sum_\alpha \bar\partial(f_\alpha)\bar\chi\big[\partial(f_\alpha)\big]$$

which are a reexpression of the completeness property.

4 Higher Order Calculus

In order to build a coherent differential calculus, we need to at least propagate the first-order algebraic relations, to second-order relations, by differentiating them. Such second-order relations need to behave in a decent way with the rest of the bundle structure, and so on and so forth. In such a way we arrive to the concept of the *universal horizontal envelope* calculus $(\mathcal{H}_\wedge, F, \mathcal{M}_\wedge)$ associated to a given first-order calculus $(\mathcal{H}, F_\wedge, d_M)$ as

explained in [4]. Let us recall that $\mathcal{H}_\wedge = \mathcal{H}^\otimes/\mathcal{J}$ where $\mathcal{J}$ is an appropriate ideal obtained as a limit of an increasing sequence of ideals, constructively started from the quadratic relations defining the universal differential envelope of $\mathcal{M}$, which are precisely the differentials of the first-order relations

$$\mathcal{S}_2 = \Big\{ \sum_{fg} d_M(f) \otimes d_M(g) \;\Big|\; \sum_{fg} f d_M(g) = 0 \Big\}. \tag{4.1}$$

We are going to carry on with this construction, taking as the input the first order calculus over a given quantum Riemann surface M we have described in the previous section.

Let us start by expressing the commutation relations between coordinate forms and $\mathcal{V}$ in terms of the bimodule $\mathcal{M}$ only. The relation between θ^* and $\mathcal{V}$ reads

$$\sum_\alpha f d_M(f_\alpha^*)\{\partial(f_\alpha)q\} = \sum_\alpha d_M(f_\alpha^*)\{\partial(f_\alpha)\epsilon(f)q\} \tag{4.2}$$

for every $q \in \mathcal{B}_1$ and $f \in \mathcal{V}$.

Remark 8. Here relaxing the hermicity assumption for the coordinate functions f_α, allowing them to be 'complex'.

By 'differentiating' the left and right side we arrive to the following quadratic relation, expressed in $\mathcal{M} \otimes_\mathcal{V} \mathcal{M}$:

$$\begin{aligned} &\sum_\alpha d_M(f) \otimes d_M(f_\alpha^*)\{\partial(f_\alpha)q\} \\ &\quad - \sum_\alpha f d_M(f_\alpha^*) \otimes d_M\{\partial(f_\alpha)q\} + \sum_\alpha d_M(f_\alpha^*) \otimes d_M\{\partial(f_\alpha)\epsilon(f)q\}. \end{aligned} \tag{4.3}$$

The above quadratic elements/relations are not all independent, in fact it is easy to see that multiplying q on left by an arbitrary $g \in \mathcal{V}$ just multiplies the entire relation by g, as terms containing $d_M(g)$ cancel due to the first-order relations. Therefore we can consider as effective only those relations containing elements g that generate the right module $\mathcal{B}_1$ and one natural choice is $g = \partial^*(f_1^*), \ldots, \partial^*(f_d^*)$ as explained in the previous section. So our

'effective' set of relations is

$$\sum_\alpha d_M(f) \otimes d_M(f_\alpha^*)\{\partial(f_\alpha)\partial^*(f_\beta^*)\} - \sum_\alpha f d_M(f_\alpha^*) \otimes d_M\{\partial(f_\alpha)\partial^*(f_\beta^*)\} + \sum_\alpha d_M(f_\alpha^*) \otimes d_M\{\partial(f_\alpha)\epsilon(f)\partial^*(f_\beta^*)\}$$

in which we can recongnize matrix components of the morphism $\varrho\colon \mathcal{V} \to \mathrm{M}_d(\mathcal{V})$ and associated projection $\Pi = \varrho(1)$. The same quadratic generators now read

$$\begin{aligned}\delta_M^\beta(f) &= \sum_\alpha d_M(f) \otimes d_M(f_\alpha \Pi_{\alpha\beta}) - \sum_\alpha d_M(f f_\alpha) \otimes d_M(\Pi_{\alpha\beta}) \\ &\qquad + \sum_\alpha d_M(f_\alpha) \otimes d_M\left[\varrho_{\alpha\beta}\epsilon(f)\right] \\ &= d_M(f) \otimes d_M^-(f_\beta) - \sum_\alpha f d_M(f_\alpha) \otimes d_M(\Pi_{\alpha\beta}) + \sum_\alpha d_M(f_\alpha) \otimes d_M\left[\varrho_{\alpha\beta}\epsilon(f)\right]\end{aligned} \tag{4.4}$$

where we have applied (3.35). There are also the corresponding *conjugate relations* associated to the commutation relations

$$\sum_\alpha \{b\epsilon(f)\partial^*(f_\alpha)\} d_M(f_\alpha) = \sum_\alpha \{b\partial^*(f_\alpha^*)\} d_M(f_\alpha) f \tag{4.5}$$

between θ and $\mathcal{V}$ where $b = q^* \in \mathcal{B}_{-1}$. They are also obtainable from (4.4) by conjugating, so we get

$$\begin{aligned}\delta_M^{\beta *}(f) &= \sum_\alpha d_M(\Pi_{\beta\alpha}) \otimes d_M(f_\alpha f) - \sum_\alpha d_M(\Pi_{\beta\alpha} f_\alpha) \otimes d_M(f) \\ &\qquad - \sum_\alpha d_M\left[\varrho_{\beta\alpha}\epsilon(f)\right] \otimes d_M(f_\alpha) \\ &= \sum_\alpha d_M(\Pi_{\beta\alpha}) \otimes d_M(f_\alpha) f - d_M^+(f_\beta) \otimes d_M(f) - \sum_\alpha d_M\left[\varrho_{\beta\alpha}\epsilon(f)\right] \otimes d_M(f_\alpha).\end{aligned} \tag{4.6}$$

It is worth mentioning that these relations correspond to the following mutually conjugate first-order relations

$$\begin{aligned}\sum_{\alpha} \varrho_{\beta\alpha}\epsilon(f)d_M(f_\alpha) &= \sum_{\alpha} \Pi_{\beta\alpha}d_M(f_\alpha)f \\ \sum_{\alpha} d_M(f_\alpha)\varrho_{\alpha\beta}\epsilon(f) &= \sum_{\alpha} f d_M(f_\alpha)\Pi_{\alpha\beta}.\end{aligned} \tag{4.7}$$

And this is how the commutation relations between $\mathcal{V}$ and coordinate forms are expressed purely in terms of $\mathcal{M}$.

Summarizing our algebraic manipulations, we have described the introduced quadratic relations as belonging to the image of maps $\delta_M^\beta \colon \mathcal{V} \to \mathcal{M} \otimes \mathcal{M}$ and their conjugates.

Lemma 5. *Interestingly, these maps are derivations coupled with the morphism ϱ, in other words*

$$\begin{aligned}\delta_M^\beta(fg) &= f\delta_M^\beta(g) + \sum_{\gamma} \delta_M^\gamma(f)\varrho_{\gamma\beta}\epsilon(g) \\ \delta_M^{\beta *}(fg) &= \sum_{\gamma} \varrho_{\beta\gamma}\epsilon(f)\delta_M^{\gamma *}(g) + \delta_M^{\gamma *}(f)g\end{aligned} \tag{4.8}$$

for every $f, g \in \mathcal{V}$.

Proof. Nothing different than a direct calculation, so

$$\begin{aligned}\delta_M^\beta(fg) = \sum_{\alpha} d_M(fg) \otimes d_M(f_\alpha\Pi_{\alpha\beta}) - \sum_{\alpha} d_M(fgf_\alpha) \otimes d_M(\Pi_{\alpha\beta}) \\ + \sum_{\alpha} d_M(f_\alpha) \otimes d_M\left[\varrho_{\alpha\beta}\epsilon(fg)\right] = \sum_{\alpha} f d_M(g) \otimes d_M(f_\alpha\Pi_{\alpha\beta}) + \\ + \sum_{\alpha} d_M(f) \otimes g d_M(f_\alpha\Pi_{\alpha\beta}) \\ - \sum_{\alpha} f d_M(gf_\alpha) \otimes d_M(\Pi_{\alpha\beta}) - \sum_{\alpha} d_M(f) \otimes g f_\alpha d_M(\Pi_{\alpha\beta}) \\ + \sum_{\alpha\gamma} d_M(f_\alpha) \otimes d_M\left[\varrho_{\alpha\gamma}\epsilon(f)\right]\varrho_{\gamma\beta}\epsilon(g) + \sum_{\alpha\gamma} d_M(f_\alpha)\varrho_{\alpha\gamma}\epsilon(f) \otimes d_M\left[\varrho_{\gamma\beta}\epsilon(g)\right]\end{aligned}$$

remembering that the tensor product is over $\mathcal{V}$. Regrouping the terms, and applying (4.7) we get

$$
\begin{aligned}
&f\delta_M^\beta(g) - f\sum_\alpha d_M(f_\alpha)\otimes d_M\left[\varrho_{\alpha\beta}\epsilon(g)\right] + \sum_\alpha d_M(f)\otimes g d_M(f_\alpha\Pi_{\alpha\beta}) \\
&\quad - \sum_\alpha d_M(f)\otimes g f_\alpha d_M(\Pi_{\alpha\beta}) + \sum_{\alpha\gamma} d_M(f_\alpha)\otimes d_M\left[\varrho_{\alpha\gamma}\epsilon(f)\right]\varrho_{\gamma\beta}\epsilon(g) \\
&+ f\sum_{\alpha\gamma} d_M(f_\alpha)\Pi_{\alpha\gamma}\otimes d_M\left[\varrho_{\gamma\beta}\epsilon(g)\right] = f\delta_M^\beta(g) - f\sum_{\alpha\gamma} d_M(f_\alpha)\otimes d_M(\Pi_{\alpha\gamma})\varrho_{\gamma\beta}\epsilon(g) \\
&+ \sum_\alpha d_M(f)\otimes g d_M(f_\alpha)\Pi_{\alpha\beta} + \sum_{\alpha\gamma} d_M(f_\alpha)\otimes d_M\left[\varrho_{\alpha\gamma}\epsilon(f)\right]\varrho_{\gamma\beta}\epsilon(g) = f\delta_M^\beta(g) \\
&\qquad + \sum_{\alpha\gamma}\Big\{ d_M(f)\otimes d_M(f_\alpha)\Pi_{\alpha\gamma} - f d_M(f_\alpha)\otimes d_M(\Pi_{\alpha\gamma}) + \\
&\qquad\qquad + d_M(f_\alpha)\otimes d_M\left[\varrho_{\alpha\gamma}\epsilon(f)\right]\Big\}\varrho_{\gamma\beta}\epsilon(g) \\
&\qquad = \sum_{\alpha\gamma}\Big\{ d_M(f)\otimes d_M(f_\alpha\Pi_{\alpha\gamma}) - d_M(f f_\alpha)\otimes d_M(\Pi_{\alpha\gamma}) + \\
&\qquad\qquad + d_M(f_\alpha)\otimes d_M\left[\varrho_{\alpha\gamma}\epsilon(f)\right]\Big\}\varrho_{\gamma\beta}\epsilon(g) \\
&\qquad\qquad + f\delta_M^\beta(g) = f\delta_M^\beta(g) + \sum_\gamma \delta_M^\gamma(f)\varrho_{\gamma\beta}\epsilon(g)
\end{aligned}
$$

which proves the first set of Leibniz rules. The second set is simply a conjugate version of the first. □

Naturally complementing these, we find another first-order relations coming from giving a canonical form to differentials $d_M(g)$ of arbitrary elements $g \in \mathcal{V}$. The formula

$$
f d_M(g) = \sum_\alpha \left\{ f\partial(g)\partial^*(f_\alpha^*)\right\} d_M(f_\alpha) + \sum_\alpha d_M(f_\alpha^*)\left\{\partial(f_\alpha)\epsilon(f)\partial^*(g)\right\} \tag{4.9}
$$

gives an explicit decomposition for generic elements of $\mathcal{M}$, and in particular pure differentials are decomposed as

$$
d_M^+(g) = \sum_\alpha \left\{\partial(g)\partial^*(f_\alpha^*)\right\} d_M(f_\alpha) \qquad d_M^-(g) = \sum_\alpha d_M(f_\alpha^*)\left\{\partial(f_\alpha)\partial^*(g)\right\} \tag{4.10}
$$

generating the following second-order relations

$$\sum_\alpha d_M\{\partial(g)\partial^*(f_\alpha^*)\}\otimes d_M(f_\alpha)\sim\sum_\alpha d_M(f_\alpha^*)\otimes d_M\{\partial(f_\alpha)\partial^*(g)\}. \quad (4.11)$$

In a certain way, the decomposition formula of differentials requires the first-order relations (4.7) defining the calculus $\mathcal{M}$. This can be seen from the Leibniz rule for d_M^+ and d_M^- as follows. We have

$$d_M^+(f_\beta f)=\sum_\alpha\{\partial(f_\beta f)\partial^*(f_\alpha^*)\}d_M(f_\alpha)=\sum_\alpha\varrho_{\beta\alpha}\epsilon(f)d_M(f_\alpha)+f_\beta d_M^+(f)$$
$$d_M^-(ff_\beta)=\sum_\alpha d_M^-(f_\alpha)\{\partial(f_\alpha)\partial^*(ff_\beta)\}=\sum_\alpha d_M(f_\alpha)\varrho_{\alpha\beta}\epsilon(f)+d_M^-(f)f_\beta$$

and hence

$$d_M^+(f_\beta)f=\sum_\alpha\varrho_{\beta\alpha}\epsilon(f)d_M(f_\alpha)\qquad fd_M^-(f_\beta)=\sum_\alpha\varrho_{\beta\alpha}\epsilon(f)d_M(f_\alpha) \quad (4.12)$$

which is equivalent to (4.7). It turns out that these relations, together with any of (4.4) or (4.6), cover the entire relation space for the universal differential envelope of $\mathcal{M}$. Let us prove this. By multiplying (4.11) by $f\in\mathcal{V}$ we obtain

$$f\sum_\alpha d_M\{\partial(g)\partial^*(f_\alpha^*)\}\otimes d_M(f_\alpha)=\sum_\alpha d_M\{f\partial(g)\partial^*(f_\alpha^*)\}\otimes d_M(f_\alpha)$$
$$-d_M(f)\otimes d_M^+(g)\sim f\sum_\alpha d_M(f_\alpha^*)\otimes d_M\{\partial(f_\alpha)\partial^*(g)\}$$

and on the other hand, the coordinate relation (4.3) for $q=\partial^*(g)$ reads

$$\sum_\alpha fd_M(f_\alpha^*)\otimes d_M\{\partial(f_\alpha)\partial^*(g)\}-d_M(f)\otimes d_M^-(g)\sim$$
$$\sim\sum_\alpha d_M(f_\alpha^*)\otimes d_M\{\partial(f_\alpha)\epsilon(f)\partial^*(g)\}.$$

By summing the above two relations, and simplifying the expression, we obtain

$$\sum_\alpha d_M\{f\partial(g)\partial^*(f_\alpha^*)\}\otimes d_M(f_\alpha) - d_M(f)\otimes d_M(g) \sim \\ \sim \sum_\alpha d_M(f_\alpha^*)\otimes d_M\{\partial(f_\alpha)\epsilon(f)\partial^*(g)\}.$$

However this covers *all* the universal differential envelope relations, because it is equivalent to 'differentiating' the generic canonical decomposition (4.9). In particular we see that every first-order relation of the form $\sum_{fg} f d_M(g) = 0$ would imply $\sum_{fg} f\partial(g) = 0$ as well as $\sum_{fg}\epsilon(f)\partial^*(g) = 0$ and hence $\sum_{fg} d_M(f)\otimes d_M(g) \sim 0$.

5 Torsion Forms and Around

The map J extends naturally as a standard derivation on the bimodule tensor algebra $\mathcal{H}^\otimes$. We shall denote by the same symbol the extension, so that

$$J(\psi_1\otimes\cdots\otimes\psi_n) = \big(J(\psi_1)\otimes\cdots\otimes\psi_n\big) + \cdots + \big(\psi_1\otimes\cdots\otimes J(\psi_n)\big) \quad (5.1)$$

with additional definition $J|\mathcal{B} = 0$. The covariance of initial J implies that the extended J is also covariant, in the sense of intertwining the action $F_\otimes\colon \mathcal{H}\to\mathcal{H}\otimes\mathcal{A}$ with itself. In particular we have $J(\mathcal{M}^\otimes)\subseteq\mathcal{M}^\otimes$. Our first compatibility condition between J and the differential structure can now be expressed as

$$J(\mathcal{S}_2)\subseteq\mathcal{S}_2. \quad (5.2)$$

Remark 9. In classical geometry the above condition is trivial, because $\mathcal{S}_2$ consists of all symmetric covariant 2-tensors. The situation changes in the general quantum context however, and in [4] we presented a simple class of counter-examples involving calculi over finite classical groups.

In order to proceed with constructing and exploring a meaningful complete calculus, and to fix ideas, we shall from now on assume that coordinate forms θ and θ^* are not only compatible with $\mathcal{V}$ representing the functions on M, but also with the entire differential calculus on the base, in the sense of *supercommuting* with all differential forms from Ω_M. Actually this slightly stronger but natural condition can be derived as a logical consequence of the existence of certain special connections. Let us elaborate on this in more detail.

As explained in [4], the supercommutant $\mathrm{Z}(\mathcal{M}_\wedge, \mathcal{H}_\wedge)$ of $\mathcal{M}_\wedge$ in $\mathcal{H}_\wedge$ possesses a natural right $\mathcal{A}$-module structure. It is given by

$$\psi \circ a = [a]_1 \psi [a]_2$$

where $[a]_1 \otimes [a]_2 = \tau(a)$ and $\tau\colon \mathcal{A} \to \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B}$ is the translation map for a quantum principal bundle P.

In this case, the following four equivalent ways describe general commutation relations

$$\begin{aligned} \psi\varphi &= (-)^{\partial\varphi\partial\psi}\varphi^{(0)}(\psi \circ \varphi^{(1)}) & \varphi\psi^* &= (-)^{\partial\varphi\partial\psi}\big(\psi^* \circ \kappa^{-1}(\varphi^{(1)})\big)\varphi^{(0)} \\ \varphi\psi &= (-)^{\partial\varphi\partial\psi}\big(\psi \circ \kappa^{-1}(\varphi^{(1)})\big)\varphi^{(0)} & \psi^*\varphi &= (-)^{\partial\varphi\partial\psi}\varphi^{(0)}(\psi^* \circ \varphi^{(1)}) \end{aligned} \tag{5.3}$$

between the forms from $\mathrm{Z}(\mathcal{M}_\wedge, \mathcal{H}_\wedge)$ and arbitrary horizontal forms φ. In particular, we can apply them for $\psi = \theta, \theta^*$.

Remark 10. Free of any fear of generating confusion, we shall use the symbol ∂ for both: the grade of homogeneous elements and the holomorphic derivative.

Remark 11. The commutativity assumption for θ and θ^*, on the other hand, implies that our initial ϵ should be trivial on $\mathcal{V}$. This also implies that on $\mathcal{V}$, both holomorphic and antiholomorphic derivatives satisfy the classical Leibniz rule.

And since the complex structure tensor J acts as a bimodule automorphism, it therefore commutes with $\circ$, which implies that holomorphic and antiholomorphic forms are preserved by $\circ$. In particular, we have

$$\theta \circ U = \mu\theta$$

where $\mu \in \mathcal{V}$ commutes with all differential forms on the base.

Remark 12. As we shall soon see, under certain geometrically reasonable assumptions, we can conclude that μ must be in fact, a real scalar!

Let us observe that the compatibility with the *-structure reads

$$\theta^* \circ U = \mu^* \theta^*.$$

And we also have

$$\theta \circ U^* = \upsilon^{-1}(\frac{1}{\mu})\theta \qquad \theta^* \circ U^* = \upsilon^{-1}(\frac{1}{\mu^*})\theta^*.$$

where υ is the $\circ$-induced automorphism of $\mathrm{Z}(\mathcal{M}_\wedge, \mathcal{H}_\wedge)$. This easily generalizes to all integer powers of U, as explained in Appendix A.

If we now combine θ and θ^* we obtain

$$\theta\theta^* = -\upsilon^{-1}(\frac{1}{\mu})\theta^*\theta \qquad \theta^*\theta = -\mu^*\theta\theta^*.$$

In order to have a consistent algebra, we see that μ must obey

$$\mu = \upsilon(\mu^*).$$

In particular, if $\mu = \mu^* = \upsilon(\mu)$ everything will work nicely. In this case the complete horizontal calculus can be taken to be of a very simple form:

$$\theta\theta^* = -\frac{1}{\mu}\theta^*\theta \quad \Leftrightarrow \quad \theta^*\theta = -\mu\theta\theta^* \tag{5.4}$$

$$\theta^2 = 0 = \theta^{*2}. \tag{5.5}$$

Remark 13. We have also assumed here that the spectrum of μ does not contain -1 in other words $1 + \mu$ is invertible in $\mathcal{V}$. The singular case of $-1 \in \mathrm{sp}(\mu)$ with its irreducible version $\mu = -1$ should be treated separately.

Definition 1. We have just described a major class of quantum Riemann surfaces, which henceforth will be called *regular Class.*

Let us also observe that

$$w_2 = \frac{i}{2}\theta\theta^* \tag{5.6}$$

is a hermitian element of Ω^2_M, corresponding to the *volume form* in the classical setting. It is a single generator of

$$\mathfrak{hor}^2_P \leftrightsquigarrow \mathcal{B}w_2 \leftrightsquigarrow w_2\mathcal{B} \tag{5.7}$$

We shall assume that the multiplication by θ is injective on differentials of functions on M. In other words

$$w\theta = 0 \wedge w = d(f) \wedge f \in \mathcal{V} \quad \Rightarrow \quad w = 0 \tag{5.8}$$

We shall also assume that our quantum surface is *connected* in the sense that

$$\ker(d_M) \cap \mathcal{V} = \mathbb{C}. \tag{5.9}$$

Remark 14. So in particular we have that $d(f)\theta = 0$ implies f must be a scalar.

Remark 15. And we can always complicate things, by considering *direct sums* of quantum frame bundles, corresponding at the geometrical level to *disjoint sums* of quantum surfaces.

Let us consider mutually conjugate ideals $\mathcal{J}^+$ and $\mathcal{J}^-$ in $\mathcal{H}_\wedge$, generated by θ and θ^* respectively. These ideals are $F_\wedge$-invariant, in the sense that $F_\wedge(\mathcal{J}^\pm) \subseteq \mathcal{J}^\pm \otimes \mathcal{A}$. Let us denote by $\mathcal{J}^+_M$ and $\mathcal{J}^-_M$ the corresponding invariant parts of these ideals.

Lemma 6. *For a generic quantum Riemann surface, the following conditions are equivalent:*

—

(*i*) *We have* $d_M(\mathcal{J}^+_M) \subset \mathcal{J}^+_M$ *which is the same as saying* $d_M(\mathcal{J}^-_M) \subseteq \mathcal{J}^-_M$.

(*ii*) *There exists a covariant derivative* $D \in \mathfrak{d}(\mathcal{H}_\wedge, F_\wedge, \mathcal{M}_\wedge)$ *such that* $D(\mathcal{J}^+) \subseteq \mathcal{J}^+$ *and* $D(\mathcal{J}^-) \subseteq \mathcal{J}^-$.

(*iii*) *The above mentioned inclusions hold for every* $D \in \mathfrak{d}(\mathcal{H}_\wedge, F_\wedge, \mathcal{M}_\wedge)$.

Proof. Actually the equivalence between (*i*) (*ii*) and (*iii*) is a consequence of a property involving general quantum horizontal calculi, as explained in more detail in [4]. Let us recall that if $\mathcal{I}$ is an F-covariant ideal in a horizontal calculus $(\mathfrak{hor}_P, F, \Omega_M)$ then $\mathcal{I} \cap \Omega_M$ is d_M-invariant iff $\mathcal{I}$ is D-invariant for some $D \in \mathfrak{d}(\mathfrak{hor}_P, F, \Omega_M)$, and in this case the invariance property holds for all covariant derivatives. □

Remark 16. It is worth noticing that the above equivalent properties are, in fact, an intrinsic property of J, since the ideals $\mathcal{J}^-$ and $\mathcal{J}^+$ are generated by eigenspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively.

Definition 2. We shall say that J is integrable iff conditions of Lemma 6 hold.

Lemma 7. *In accordance with the classical tradition: For a regular class quantum Riemann surface, its complex structure tensor is always integrable.*

Proof. The module of horizontal 2-forms is generated by a single volume form w_2 and therefore both $D(\theta)$ and $D(\theta^*)$ can be written as a multiple of w_2 and some 'function' from $\mathcal{B}$. □

The complex structure extends to the entire algebra of horizontal forms, by means of a standard derivation. We shall use the same symbol $J\colon \mathcal{H}_\wedge \to \mathcal{H}_\wedge$ to denote such an extension. Let us also introduce a one-parameter group of automorphisms of $\mathcal{H}$ given by $j_t = \exp(tJ) = cos(t) + sin(t)J$. If $D\colon \mathcal{H}_\wedge \to \mathcal{H}_\wedge$ is an arbitrary covariant derivative then $j_{-t}Dj_t$ is also a valid covariant derivative, and moreover

$$\frac{1}{2\pi}\int_0^{2\pi} j_{-t}Dj_t dt = \frac{1}{2}(D - JDJ)$$

is the projection of D to its invariant part. Note that $j_{\pi/2} = J$ so that the above projection is just the affine mean between covariant derivatives D and $-JDJ$.

Therefore we can always assume that D is compatible with the complex structure, in the sense of commuting with J, and this will be our assumption from now on.

Remark 17. Actually we have to prove a little bit more, namely that $-JDJ$ is also a hermitian covariant derivative in the sense of [4]. But this follows straightforwardly from the fact that j_t are *-automorphisms.

Definition 3. Let us recall, that for a given covariant derivative D, its torsion is given by $\Theta = D(\theta)$.

Lemma 8. *In a regular class quantum Riemann surface, we can always find such a covariant derivative D so that*

$$D(\theta) = 0 = D(\theta^*). \tag{5.10}$$

In other words, such a connection D is torsionless.

Proof. Let us commence with an arbitrary regular covariant derivative, the torsion of which is not necessarily zero. In any case, we have

$$D(\theta) = bw_2 = \frac{i}{2} b\theta\theta^*,$$

where $b \in \mathcal{B}_1$ commutes with the base calculus. Now we can construct a displacement map $\lambda\colon \mathcal{A} \to \mathfrak{hor}_P^1$, generating a displacement tensor S, so that

$$S(\varphi) = (-)^{\partial\varphi\partial\lambda}\varphi^{(0)}\lambda(\varphi^{(1)})$$

and such that $D + S$ is torsionless.

If we define the map λ only on U, then it can be extended, by $\circ$-Leibniz rule to the whole $\mathcal{A}$. It is worth noticing that $\lambda(a)^* = -\lambda[\kappa(a)^*]$ for every $a \in \mathcal{A}$. This property is equivalent to hermicity of S. For generators U^n it means that all $\lambda(U^n)$ are antihermitian (it is sufficient to have it for $\lambda(U)$).

So, everything works fine if we define

$$\lambda(U) = \frac{1}{\mu}\frac{i}{2}(b\theta^* + \theta b).$$

Indeed, in this case

$$(D + S)(\theta) = D(\theta) - \theta\lambda(U) = \frac{i}{2} b\theta\theta^* - \frac{1}{\mu}\frac{i}{2}\theta b\theta^* = 0$$

and $D + S$ is a hermitian covariant antiderivation on $\mathfrak{hor}_P$. □

Proposition 9. *If M is a connected quantum Riemann surface and D a covariant derivative with zero torsion then*

(i) From the fact that the coordinate forms θ and θ^* commute with the elements of $\mathcal{V}$, it follows that they necessarily supercommute with the elements of $\mathcal{M}_\wedge$.

(ii) The central element μ determining the $\circ$ structure on θ, θ^* is a non-zero real scalar.

(iii) For $\mu \neq -1$ the algebra $\mathcal{H}_\wedge$ is defined by the relations

$$\begin{gathered} \theta b = b\mu^k \theta \quad \theta^* b = \mu^{-k} b\theta^* \qquad \forall b \in \mathcal{B}_k \\ \theta^2 = \theta^{*2} = 0 \\ \theta^*\theta = -\mu\theta\theta^* \end{gathered} \tag{5.11}$$

and the special case $\mu = -1$ leads to $\mathcal{H}_\wedge$ generated by

$$\begin{gathered} \theta\theta^* = \theta^*\theta \\ \theta b = b(-)^k\theta \quad \theta^* b = (-)^k b\theta^* \end{gathered} \tag{5.12}$$

for every $b \in \mathcal{B}_k$.

Proof. Let us assume that D is torsionless, that is, $D(\theta) = D(\theta^*) = 0$. This implies, together with the Leibniz rule for D, that D anticommutes with all elements of the form $D(f)$ where $f \in \mathcal{V}$. Since the algebra $\mathcal{M}_\wedge$ is generated by elements f and $D(f)$ it follows that θ supercommutes with all elements of $\mathcal{M}_\wedge$. This is quite a non-trivial condition for θ. Furthermore, since D commutes with the $\circ$-structure on $\mathrm{Z}(\mathcal{M}_\wedge, \mathcal{H}_\wedge)$ we obtain

$$D(\theta \circ U) = D(\mu)\theta = D(\theta) \circ U = 0 \tag{5.13}$$

and it follows that $D(\mu) = 0$. Because of the connectedness of M we conclude that μ is a (non-zero) scalar.

Now, because of θ supercommuting with the elements of $\mathcal{M}_\wedge$, we have

$$\theta\varphi = (-)^{\partial\varphi}\varphi^{(0)}(\theta \circ \varphi^{(1)}) \qquad \forall \varphi \in \mathcal{H}_\wedge \tag{5.14}$$

and this implies

$$\theta^2 = -\mu\theta^2 \qquad \theta^*\theta = -\mu^*\theta\theta^*$$

so if $\mu \neq -1$ we conclude that squares of θ and θ^* vanish. Conjugating the second relation above, it follows that $\mu = \mu^*$ that is μ must be a real non-zero scalar. □

Proposition 10. *The torsion Θ with the differential $d_M \colon \mathcal{M}_\wedge \to \mathcal{M}_\wedge$ determines completely the covariant derivative D. In particular, if there exists a torsion-free covariant derivative, then such a covariant derivative is unique.* □

6 Concluding Observations

The context of quantum Riemann surfaces provides a rich supply of examples of quantum principal frame bundles, exhibiting an array of purely quantum phenomena. Including a fundamental importance of the quantum geometrical context. The standard quantum Hopf fibration already illustrates how a quantum nature of the bundle, naturally propagates to the classical structure group SO(2) in the form of a non-classical induced differential calculus. This calculus can in turn be used as a toy model, to investigate properties of geometrically interesting connections, as for example strong [12] or regular [3] connections. In this paper we have shown that these phenomena are in fact a general feature of (almost) all quantum Riemann surfaces, and established a general playground for their computations and study. Spinorial structures, and in particular diagrammatical ideas developed in [14], can be also naturally reflected and formulated within this framework.

A very interesting class of illustrative examples of quantum Riemann surfaces can be constructed in the spirit of a stochastic version of quantum geometry, developed and presented in detail in [16]. In stochastic geometry, we interpret a classical representation of the space continuum (as a smooth manifold equipped with an atlas) as a natural way to give an operational meaning to the mean values of quantum coordinates and related quantum observables. So a 'point' is actually behaving as a kind of a 'coherent state' on the 'true quantum algebra' representing the quantum underlying space. Hilbert bundles for representing field configurations, as geometric excitations, naturally appear if we apply the GNS construction to these states-points. If this stochastic geometric interpretation preserves some classical symmetries of the initial classical space, they will naturally induce automorphisms of the entire quantum frame bundle, corresponding to the quantum underlying Riemann surface. In such a way, we obtain interesting stochastic versions of all classical surfaces of constant curvature.

Basic examples of this kind are quantum Euclidean plane, given by the relations

$$pq - qp = \frac{\hbar}{i}$$

with hermitian p and q, and its $\mathbb{Z} \times \mathbb{Z}$ quotient, the quantum torus, given by the unitaries U and V and a single relation $VU = zUV$. Here $\hbar > 0$ and $|z| = 1$.

On the other hand, the Toepliz extension, briefly featured in Appendix B provides a natural background to develop an example of a quantum hyperbolic plane.

The plane retains the classical symmetry group $\mathrm{SU}(1,1)/\{-1,1\}$ given by the hyperbolic 2×2 matrices acting on the adjoint shift operator $z = S^*$, in the classical way:

$$z \rightsquigarrow \frac{az+b}{\bar{b}z+\bar{d}}.$$

This operator plays the role of the 'complex variable' for the quantum Poincaré model. Quantum versions of all higher genus $g \geq 2$ surfaces can be constructed as appropriate factor spaces of this model, just like in the classical theory. In particular, a quantum version of Klein quartic surface lives there.

All these beautiful examples nicely fit in the presented conceptual framework. We shall study them in the second and third part of this essay, where we shall also present a complete integration theory, a quantum version of the Gauss-Bonnet theorem, and analyze in detail the Euler class, and other characteristic classes of quantum Riemann surfaces.

A On Complex Structure and Derivatives

In this appendix we have collected a number of simple but important and useful identities used in the calculations involving coordinate forms θ and θ^*.

The bimodule property implies that

$$J(\varphi \circ a) = J(\varphi) \circ a \tag{A.1}$$

for every $\varphi \in \mathcal{H}$ and $a \in \mathcal{A}$. Indeed, we have

$$J(\varphi \circ a) = J\{[a]_1 \varphi [a]_2\} = [a]_1 J(\varphi)[a]_2 = J(\varphi) \circ a.$$

As we have already mentioned the $\circ$ structure commutes with J, so it it follows that

$$\theta \circ U = \mu\theta = \theta\mu \tag{A.2}$$

where μ is a central element of $\mathcal{V}$. And taking into account that circling with U commutes with the *-structure, we conclude

$$\theta^* \circ U = \mu^*\theta^* = \theta^*\mu^*. \tag{A.3}$$

In general, the circling with U is giving a nice *-automorphism of the corresponding commutant space, as $\mathrm{Z}(\mathcal{M}_\wedge, \mathcal{H}_\wedge)$.

Let us consider this, in a general algebraic context. We can compute the action of all positive and negative powers of U on θ and θ^*. Iterating the above equalities, we obtain

$$\begin{aligned} \theta \circ U^{n+1} &= \Big\{\prod_{k=0}^{n} \upsilon^{n-k}(\mu)\Big\}\theta \qquad & \theta^* \circ U^{n+1} &= \Big\{\prod_{k=0}^{n} \upsilon^{n-k}(\mu^*)\Big\}\theta^* \\ \theta \circ U^{-n} &= \Big\{\prod_{k=1}^{n} \upsilon^{-k}(\frac{1}{\mu})\Big\}\theta \qquad & \theta^* \circ U^{-n} &= \Big\{\prod_{k=1}^{n} \upsilon^{-k}(\frac{1}{\mu^*})\Big\}\theta^* \end{aligned} \tag{A.4}$$

where υ is precisely this 'circling' automorphism.

We have applied a number of general properties of covariant derivatives. At first, as the element of the vector space of the affine space of covariant derivatives, the map S should be a covariant first-order *-antiderivation linear over $\mathcal{M}_\wedge$. From the general theory of quantum frame bundles, we know that such maps are in a one-one correspondence with functionals $\lambda\colon \mathcal{A} \to \mathrm{Z}(\mathcal{M}_\wedge, \mathcal{H}_\wedge)$ satisfying

$$\lambda(\mathcal{A}) \subseteq \mathcal{M}^1_\wedge \cap \mathrm{Z}(\mathcal{M}_\wedge, \mathcal{H}_\wedge) \tag{A.5}$$

$$\lambda(ab) = \epsilon(a)\lambda(b) + \lambda(a) \circ b \tag{A.6}$$

$$\lambda(a)^* = -\lambda[\kappa(a)^*]. \tag{A.7}$$

Remark 18. The first of these equations is a consequence of a general property, that such maps λ intertwine coactions ad, the adjoint one $\mathrm{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)})a^{(3)}$. and $F_\wedge$. In our case the adjoint action is trivial, as the group is Abelian.

A couple of words regarding the displacement operators. As we have already recalled, there is an intrinsic connection between S and λ. It is given by

$$S(\varphi) = (-)^{\partial\varphi\partial\lambda}\varphi^{(0)}\lambda(\varphi^{(1)}) \qquad \lambda(a) = [a]_1 S([a]_2).$$

We see that the map S is determined by a single value $\lambda(U)$ which is a supercentral antihermitian one-form on M.

Finally, let us consider a general covariant derivative D. As it extends d_M and is covariant, it preserves the subalgebra $\mathrm{Z}(\mathcal{M}_\wedge, \mathcal{H}_\wedge)$. A very nice property is that D always commutes with the $\circ$-structure on $\mathrm{Z}(\mathcal{M}_\wedge, \mathcal{H}_\wedge)$. Indeed, we have

$$\begin{aligned} D(\varphi\circ a) &= D([a]_1\varphi[a]_2) = D([a]_1)\varphi[a]_2 + [a]_1 D(\varphi)[a]_2 + (-1)^{\partial\varphi}[a]_1\varphi D([a]_2) \\ &= D([a^{(1)}]_1)[a^{(1)}]_2(\varphi\circ a^{(2)}) + D(\varphi)\circ a + [a^{(1)}]_1 D([a^{(1)}]_2)(\varphi\circ a^{(2)}) \\ &= D([a^{(1)}]_1[a^{(1)}]_2)(\varphi\circ a^{(2)}) + D(\varphi)\circ a = D(\varphi)\circ a. \end{aligned}$$

We have used the covariance of D, and the translation map. We have also applied the 'scalar reduction' property $[a]_1[a]_2 = \epsilon(a)1$ of the translation map.

B Freeness Property & Toepliz Extension

In this appendix we shall prove that the natural action of SO(2) on the Toepliz extension algebra is not free.

Let $H = l^2(\mathbb{N})$ be the Hilbert space of square-sumable complex sequences on $\mathbb{N}$ and consider the shift operator $S\colon H \to H$ given by

$$Se_k = e_{k+1}$$

where $(e_k)_{k\in\mathbb{N}}$ are the canonical orthonormal base in H. The adjoint operator is given by

$$S^*e_1 = 0 \qquad S^*e_k = e_{k-1} \qquad k \geq 2.$$

Let $\mathcal{T}$ be the C*-algebra generated by S. It contains the whole ideal $\mathbb{K}$ of compact operators in H, the factor-algebra is commutative and naturally identificable with functions on a circle (commutative C*-algebra generated by a single unitary element and no further relations) and we have a natural extension

$$0 \to \mathbb{K} \hookrightarrow \mathcal{T} \to C[\mathrm{SO}(2)] \to 0.$$

The ideal $\mathbb{K}$ in $\mathcal{T}$ coincides with the *commutant* of $\mathcal{T}$–the minimal ideal containing all the commutators. The same algebra $\mathcal{T}$ can be defined in abstract terms, as generated by a single generator S and a single relation $S^*S = 1$. Let $\mathcal{B}$ be the everywhere dense *-subalgebra of $\mathcal{T}$ generated by S. The group $G = \mathrm{SO}(2)$ naturally acts by automorphisms of $\mathcal{T}$, preserving $\mathcal{B}$, via $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ given by $F(S) = S \otimes U$ and we have natural decompositions

$$\mathcal{T} = \oplus_{k\in\mathbb{Z}} \mathcal{T}_k \qquad \mathcal{B} = \oplus_{k\in\mathbb{Z}} \mathcal{B}_k$$

with $\mathcal{B}_k$ dense in $\mathcal{T}_k$. The elements of the form $b_{lk} = S^l S^{*k}$ constitute a linear basis in $\mathcal{B}$. In particular the elements $f_d = b_{dd}$ for $d \in \mathbb{N}$ form a sequence of decreasing orthogonal projectors, and together with $f_0 = b_{00} = 1$ span a commutative algebra $\mathcal{V} = \mathcal{B}_0$. We have

$$f_n f_m = f_{\max(m,n)}. \tag{B.1}$$

In terms of the Hilbert space realization, the algebra $\mathcal{T}_0$ consists of all diagonal (infinite) matrices of $\mathcal{T}$. The Gelfand-Neimark spectrum of this commutative C*-algebra is homeomorphic to the space $\infty \cup \mathbb{N}$ the one-point compactification of $\mathbb{N}$ in its discrete topology. The characters κ_n corresponding to natural numbers n are realized via the common eigenvectors e_n of $\mathcal{T}_0$, in other words $fe_n = \kappa_n(f)e_n$. It is also worth mentioning that

$$\mathcal{T}_k/(\mathcal{T}_k \cap \mathbb{K}) \leftrightarrow \mathbb{C}U^k.$$

Lemma 11. (*i*) *All the elements of the space $\mathcal{T}_{-1}$ are of the form fS^* for $f \in \mathcal{T}_0$.*

(*ii*) *The following commutation properties hold*

$$bS^* = S^*\{SbS^*\} \qquad S^*f = \{S^*fS\}S^*$$

for $b \in \mathcal{T}$ and $f \in \mathcal{T}_0$.

Proof. It is clear that (i) is true for elements of $\mathcal{B}_{-1}$. Now consider an arbitrary $b \in \mathcal{T}_{-1}$ and a sequence of elements from $\mathcal{B}_{-1}$ converging to it: $f_q S^* \to b$. Multiplying by S on the right we obtain $f_q \to bS = f \in \mathcal{V}$ and hence

$$f_q S^* \to bSS^* = b = fS^*.$$

We see that $f = bS$ is uniquely determined by b.

As for property (ii) the first equation is an obvious consequence of the generating relation for $\mathcal{T}$. The second equation holds for scalars, and the projectors f_d, indeed

$$\{S^* f_d S\} S^* = S^* S^d S^{*d} S S^* = S^* S^d S^{*(d-1)} S^* = S^* f_d.$$

Therefore, by linearity it holds for all $f \in \mathcal{V}$. Now taking the closure, we conclude that it holds for all $f \in \mathcal{T}_0$. □

Remark 19. By the same reasoning we see that more generally, the elements of the space $\mathcal{T}_{-n}$ are expressable in the form $b = fS^{*n}$.

Lemma 12. *Let $(q_\alpha)_{\alpha \in \Lambda}$ be a finite family of elements of $\mathcal{T}_{-1}$. Then the expression $f = \sum_\alpha q_\alpha^* q_\alpha$ is never invertible.*

Proof. By Lemma 11 we can write $q_\alpha = f_\alpha S^*$ for some $f_\alpha \in \mathcal{T}_0$ and therefore

$$f = \sum_\alpha S f_\alpha^* f_\alpha S^*.$$

Such an operator always has a non-trivial kernel containing e_1 and consequently it can not be invertible. □

We now conclude that the action F is not free, altough one of the two freeness equations (2.3) is fulfilled, as we can consider for example the generators S and S^*. A modification and generalization of this construction leads to Cuntz algebras (where both conditions are elegantly satisfied) establishing a framework for a rich class of quantum Riemann surfaces.

However, in a truly geometric theory [5], as we shall see, there is only one freeness condition, as all relevant maps are *-homomorphism. Within this formulation, the same action will be free. This property has a nice

geometrical interpretation. Namely, the Toepliz extension algebra can be naturally interpreted as a non-commutative playground for quantum hyperbolic plane. The introduced rotations are nothing but rotations around the origin-zero of the stochastic Poincaré model. Classically, the violation of the freeness occurs only at 0. But the model is 'pointless' and there is no any origin here, so the action will be free.

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GEOMETRIC QUANTUM GROUPS AS REALIZATIONS OF DIAGRAMMATIC SYMMETRY CATEGORY
Extended Version 2

Micho Đurđevich
Instituto de Matemáticas, UNAM
Area de la Investigación Científica
Circuito Exterior, Ciudad Universitaria
México DF, CP 04510, México
micho@matem.unam.mx
www.matem.unam.mx/~micho

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Abstract

The paper is a complement to a general diagrammatic formulation of the concept of symmetry. Important general and special diagrammatic identities are discussed. We present a concrete non-commutative geometric version of the formalism suitable for categories of quantum spaces, groups and bundles. We follow the basic philosophy of the formalism of diagrammatic symmetries, by considering functors from the primary category, where objects are natural numbers and morphisms are diagrams generated by the archetypical product and coproduct, into a category of quantum spaces (represented by appropriate non-commutative *-algebras and their homomorphisms). Illustrative examples are presented.

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1 Introduction

In exploring different mathematical structures, and their applications in theoretical physics, the concept of symmetry is proven to be one of those fundamental conceptual nodes, that allows us to simplify, interconnect, and see a deeper structure, often hidden behind a variety of details, appearing perhaps to a first sight as something important, and which nevertheless turn out to be irrelevant.

The idea of symmetry transcends concreteness of mathematical realms. Therefore, in trying to understand what symmetry is as such, we should use a language which is capable of extracting the essence from a variety of specific situations. One possible such a way in formulating the concept of symmetry, is given by *diagrammatic categories*. We can introduce a primary diagrammatic category, representing the very essence of symmetry. As such, it must be something very simple. Indeed, its objects are just natural numbers, and morphisms certain diagrams connecting the numbers. Any concrete realization of symmetry, is then interpreted as a functor from this category, into a given category of our mathematical discourse.

The diagrammatic technique illustrates the importance of language. Inadequate languages distort the truth, and make things artificially complicated. The truth is revealed in its full simplicity and harmony, only when appropriately described.

The aim of this study is twofold. At first, we shall make an overview of a diagrammatic formulation of symmetry, complementary to a sketchy exposition of [1]. Here, along the general properties, we shall focus on subtle interrelations between different diagrammatical constructs. Secondly, we shall look at how the diagrammatic formulation projects into the realm of quantum spaces. Our primary interest is to define a concept of group for quantum spaces. It turns out that a consistent projection from diagrams to quantum spaces, naturally promotes an interesting refinement of the very concept of internal geometry of a space–be it classical or quantum. We need to take into account possible 'collectivity modes' of geometry, as a part of the internal geometry of a given individual space. In subsequent publications, we shall unify these ideas with quantum principal bundles [2, 3, 4, 13] where such quantum groups play the role of structure groups,

with the base manifold and bundle being the appropriate quantum objets.

This is the second enlarged version of the original manuscript written in 2014. It is organized in the following way. In the next section, we outline the principal steps for a general diagrammatical formulation of symmetry. This formulation is very general, in that applies to all mathematical universes, where there exists a possibility to perform certain elementary operations. In all such realms, we can introduce the concept of symmetry in a natural, and internal way. As such, the diagrammatic formulation is free of any contents. Our general consideration will be followed by a complementary discussion of several important properties, which further sharpen our vision of the mathematical category in which the symmetries are operating. These additional conditions will be fulfilled in the category of quantum spaces, in which we are principally interested here.

In Section 3 an algebraic (and in particular a C*-algebraic) framework for our formulations is defined and studied. The key idea is to introduce the concept of *internal collectivity* for a given quantum space. This collectivity structure is represented as a series of suitable ideals $J_n$in the n-fold free C*-products of A with itself. The corresponding factor algebras A_n are interpreted as suitable 'collectivity modes' for the quantum space, and are thought of as an inherent part of its geometry. The product maps (corresponding at the geometrical level to the diagonal map) naturally project to the factor algebras A_n. When we consider the coproduct, it would be a morphism from A to A_2. The entire diagrammatic category is then realizable within the collection of C*-algebras A_n, via a functor assigning $n \mapsto A_n$ and the corresponding combinations of the product and coproduct morphisms, to the diagrams.

It is important to emphasise that all morphisms in our formulation are unital *-algebra homomorphisms. This is the principal difference to the Hopf algebras, which turn out to be inadequate to properly represent group structures on quantum spaces. We can include in our theory objects that are not describable in terms of Hopf algebras, like for example quantum spaces without any points. In our theory, the unit and counit appear as a kind of a unified entity, which we shall call *co-unit.*

It is also worth emphasizing that in our diagrammatic formulation, the co-unit as well as the antipode morphism, are explicitly expressed in terms

of the two primordial morphisms, corresponding to the diagonal map, and the product map, respectively.

In Section 4 a collection of interesting and illustrative examples are presented. We start by explaining how the free groups are included in the diagrammatic framework, and proving a couple of interesting facts. By considering the simplest factorization, that of $\mathbb{Z}$ over the cyclic group $\mathbb{Z}_n$, at the one-particle level, we show that free Burnside groups $B(d,n)$ naturally appear at the level of d-collectivity. The finite universal Burnside groups are naturally includable, too.

We then discuss in detail the universal unitary groups, and a more general formulation of a compact matrix quantum group. We shall observe that, by appropriately redefining the collectivity algebras, it is possible to naturally include the idea of an internal homogeneity of a group quantum space, by allowing the 'neutral element/transformation' to be unspecified, reflecting the classical truth that any element can be considered as a neutral element, for appropriately shifted products.

As an interesting topic in its own light, we shall briefly consider some non-standard set-theoretic realizations, and a braided version of quantum SU(2), corresponding to a unitary deformation, in the sense of allowing unitary central group-like objects playing the role of 'deformation' parameters. This is realized via a simple cross-product type construction. We shall also present a 'fluctuating co-unit' version of a quantum SU(2) group, which allows us to consider the deformation parameter μ as a complex number (and not as an 'active' central element).

We also present the quantum torus, with a natural quantum group structure on it. We consider two different realizations: the one with the deformation parameter as above mentioned 'active' central element, and the standard quantum torus where the deformation parameter is a unitary complex number. The former leads to a standard quantum group, and the latter is an example of a fluctuating co-unit structure. Related to the quantum torus are quantum Euclidean plane (and more generally Heisenberg-Weyl algebras) and Clifford algebras. We briefly describe how universal envelopes of Lie algebras are included in the theory.

Another class of interesting examples comes from quantum differential calculi. We shall discuss here these structures built over finite groups. It

turns out that if a first-order calculus Γ over a finite group G is associated to an invariant subgroup (or more generally, to a union of invariant subgroups), then there exists a natural quantum collectivity system for the graded differential *-algebra $\Gamma^\wedge$, the universal differential envelope of Γ. It is based on 'multi-particle' iterations of Γ, as quantum differential calculi over direct products of G with itself.

The paper concludes with three appendices, having an independent interest. We shall compute the simplest non-commutative cyclic group, the 27-element $B(2,3)$. This is the universal collectivity for the cyclic group $\mathbb{Z}_3$ at the level of 'quantum pairs'. In the second appendix we have collected some calculations related to quantum determinants and unitary triples related to quantum U(2) group. And finally, in the last appendix we describe the cross-products, used in the construction of the 'active' μ quantum SU(2) group.

Acknowledgement: I am very indebted to Prof. Zbigniew Oziewicz for numerous discussions and constructive remarks, and for his constant encouragements and enthusiastic support during the preparation of these 3 manuscripts.

2 Diagrammatic Formulation

2.1 General Properties

Our diagrammatic category is based on two fundamental transformations symbolically represented as

(2.1)

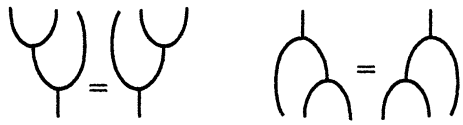
the primary morphisms between One and 2. The objects of our category are natural numbers, and morphisms are diagrams constructed from these two primary transformations. We shall imagine that Platonic 'time' flows from up to down. The first morphism can be interpreted as the duplication

movement, that is, the diagonal map. The second morphism is the composition: Two symmetry forms are transformed into a new one, by composing them. So if we consider say a standard group G, the first morphism is from G to $G \times G$, and does $x \mapsto (x, x)$. The second morphism transforms $G \times G$ into G, via the group composition $(x, y) \mapsto xy$.

A dual version, which is of our principal interest in this essay, interprets the first morphism as the coproduct, the dual of the composition map. The second morphism is then interpreted as the diagonal map. The simplest realization of the dual version, is via Hopf algebras $\mathcal{A}$. In this case the product is simply the tensor product $\otimes$, diagonal map is the product $m\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$, and the composition is given by the coproduct $\phi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Our quantum geometrical formulation will be based on appropriate in general non-commutative *-algebras, however we shall not deal with Hopf algebras as such (except in the classical case of commutative algebras, and other related special scenarios as superalgebras and involutive braided-commutative structures).

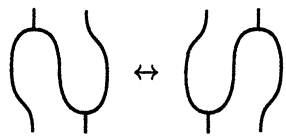
As already mentioned, our diagrams can be viewed as morphisms in a symbolic category, the objects of which are positive natural numbers. Every diagram-morphism between numbers n and m possesses n upper 'entry' legs and m lower 'exit' legs. Composing two morphisms simply means connecting the exit legs of the first with the entry legs of the second.

In accordance with our principal interpretation, we shall assume that the following co/associativity properties hold:



(2.2)

We shall also postulate that the following mutually mirrored combinations of our principal morphisms



(2.3)

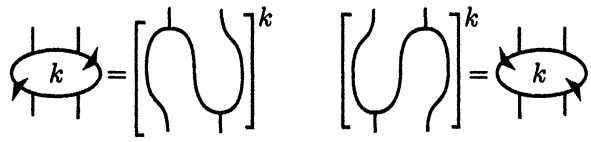
are *invertible* in our symbolic category. So both are automorphisms of the object number 2.

Our entire category is constructed by combining these two primordial morphisms. We have two principal construction movements: the composition and the product. The product of the morphisms is diagrammatically expressed by simply putting the factors side-by-side.

We shall also assume that the category is such that the *cancellation property* holds. If either the left or right direct products of two morphisms with the identity morphism on the initial object number One are equal, then the morphisms in question must be equal. We can interpret this property as the third construction movement–a recursive way to produce new identities from the already established ones. There is something contextual in it.

Remark 2.1. Our diagrammatic category is, by definition, symmetric under left-right and up-down flips—so the Klein symmetry group happily acts here.

It turns out to be both natural and useful, to separately consider the following 'elliptical' expressions



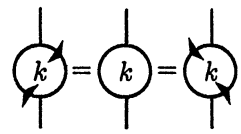
(2.4)

where the above mentioned automorphisms of 2 are elevated to arbitrary integer powers $k \in \mathbb{Z}$. Associated to these morphisms are the following 'circular' combinations

(2.5)

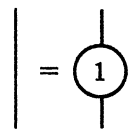
for every $k \in \mathbb{Z}$. It turns out that, although the definition of the elliptic morphisms exhibits an inherent orientation, the circular morphisms are

actually non-oriented. In other words,



(2.6)

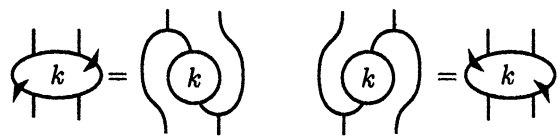
precisely defining here the non-oriented circular morphisms, for every $k \in \mathbb{Z}$. It can easily be shown that



(2.7)

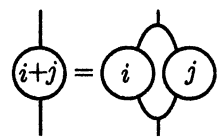
in other words, the circular morphism corresponding to $k = 1$ is the identical transformation of 1, the initial object.

The elliptic morphisms can be reverse expressed in terms of the circular morphisms in the following way



(2.8)

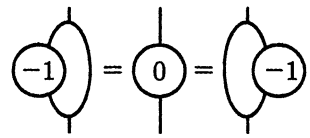
and the circular morphisms satisfy the following interesting convolution property



(2.9)

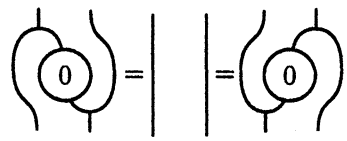
for every $i, j \in \mathbb{Z}$. In particular, all these morphisms with $k > 0$ can be generated by iterating the convolution composition of the identical morphism ($k = 1$) with itself, and similarly the negative-indexed morphisms are iterations of the antipodal morphism corresponding to $k = -1$. We shall use the symbols c_k to refer to the circular morphisms, in either diagrammatic form or some concrete realization.

The following antipodal identity



(2.10)

a particular case of the convolution property, is of a very special interest. It features the zero-morphism, the circular morphism indexed by $k = 0$, which also acts as a 'phantom particle' according to



(2.11)

which is a special case of the above elliptic-circular connection.

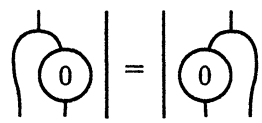
It follows that every morphism in our diagrammatic category is constructible by combining the two primary morphisms, and the antipodal morphism c_{-1}.

In our basic example of classical groups, this morphism maps every element of G into the neutral element. The antipodal morphism corresponds to the transformation $x \mapsto x^{-1}$. In the Hopf algebras scenario, we have

$$c_0 = 1\epsilon \qquad c_{-1} = \kappa \tag{2.12}$$

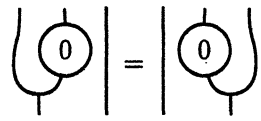
where $\epsilon\colon \mathcal{A} \to \mathbb{C}$ and $\kappa\colon \mathcal{A} \to \mathcal{A}$ are the standard counit and the antipode respectively. For this reason we shall call the morphism c_0, the *co-unit.* Emphasizing the unified nature of the concept. Also, c_{-1} will be called the *antipode.*

The following identities involving the co-unit morphism, complete the set of its fundamental identities. At first, we always have



(2.13)

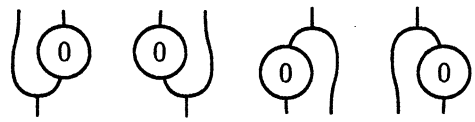
We can intuitively think of this property as a kind of tensoriality over the base number field. Or, at the dual level



(2.14)

as a kind of elementary fibration-type nature of the direct product. Both identities easily follow by considering a wave-like form from 2 to 3, or 3 to 2, with the co-unit on the second and third positions in the middle of the diagram.

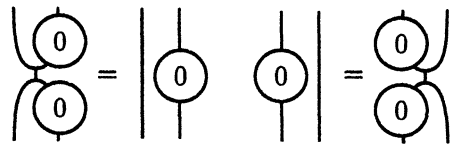
The four morphisms



(2.15)

can be viewed as elementary 'projection' transformations from 2 to One and 'insertion' transformations, from One to 2. They can be naturally combined to create higher-order 'insertions' and 'projections'.

The second fundamental property is



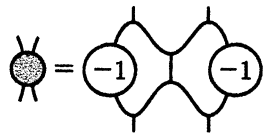
(2.16)

We shall call it the *continuability* condition. It can be used as a basis for an alternative formulation of the theory, where we start from the primary morphisms, and postulate the existence of c_0 and c_{-1} satisfying their convolution properties. Both identities easily follow by adding a free line to the left, or to the right, respectively, and playing with the tensoriality and phantom particle property.

So if in addition the continuability holds, then the theory turns out to be equivalent to our initial formulation. The advantage of such a formulation is that it does not postulate the invertibility nor cancelability properties.

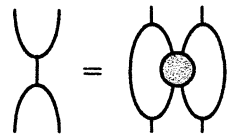
They all follow as a consequence of the initial diagrammatic identities. In particular, the continuability ensures that the general left and right convolutions with the co-unit behave functorially with respect to the composition of morphisms, and hence it is easy to deduce that all such convolutions are actually trivial. This is a primary property of the co-unit which in its turn, implies the left and right cancelability in general.

The antipode morphism, together with the principal product/coproduct morphisms, combines in a very interesting way to produce the canonical *flip-over* morphism, represented here by a smaller grey disk:



(2.17)

The above property is equivalent to requiring



(2.18)

which is a kind of multiplicativity condition for the coproduct map. In the classical case this morphism is just the standard flip. We can see this by performing simple substitutions in the diagram definition. For the standard groups we have

$$(x,y) \mapsto (x,x,y,y) \mapsto (x^{-1},xy,y^{-1}) \mapsto (x^{-1},xy,xy,y^{-1}) \mapsto$$
$$\mapsto (x^{-1}(xy),(xy)y^{-1}) = (y,x).$$

And we also have, at the dual level of Hopf algebra realizations

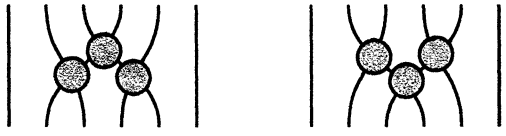
$$a \otimes b \mapsto a^{(1)} \otimes a^{(2)} \otimes b^{(1)} \otimes b^{(2)} \mapsto \kappa(a^{(1)})\phi(a^{(2)}b^{(1)})\kappa(b^{(2)}) =$$
$$\kappa(a^{(1)})(a^{(2)}b^{(1)} \otimes a^{(3)}b^{(2)})\kappa(b^{(3)}) = \epsilon(a^{(1)})b^{(1)} \otimes a^{(2)}\epsilon(b^{(2)}) = b \otimes a.$$

We shall use a truly geometric realization of the diagrammatic category, to formulate a concept of the quantum group. This realization is different, in a

number of important ways, from the standard Hopf-algebra formulation. It promotes an array of special identities, which are not fulfilled in the general diagrammatic theory. We shall now study these conditions in their pure diagrammatic settings.

2.2 Special Conditions

For every integer $n \geq 2$ there exists a natural product morphism $\mu_n \colon n+n \to n$. It is constructed by naturally representing the 'shuffle' permutation on $2n$, which brings the $(n+k)$-th entry right next to the k-th. The shuffle is composition of $n(n-1)/2$ flips. In a similar way, we can define the corresponding anti shuffling diagrams. For example, in the case $n = 3$ the shuffle and antishuffle are given by



(2.19)

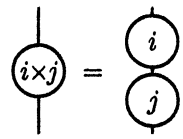
respectively.

The product morphism is then given by composing the shuffle with the n-fold direct product of diagonal map. Similarly, we can construct a coproduct morphism $\phi_n \colon n \to n+n$. We would have to compose the n-fold coproduct with the corresponding antishuffling morphism, mirroring the construction of the product. We can think of these morphisms as 'multi-particle' counterparts of the primary diagonal and composition transformations. In general they will not satisfy their corresponding co/associativity property.

For us here, the product structure will be conceptually more important. The property (2.18), which as we have mentioned defines the flip-over morphism, states that the composition morphism intertwines the product structures on 1 and 2.

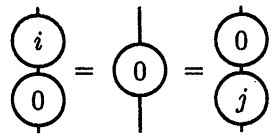
Our first special condition is a counterpart of the convolution property for circular morphisms, and provides the ring structure of integers, within

diagrams. It says that $c_i c_j = c_{ij}$, or diagrammatically



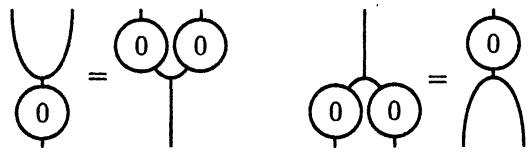
(2.20)

for every $i, j \in \mathbb{Z}$. In particular, in this case the antipodal morphism c_{-1} is involutive. It is also worth noticing that the multiplicativity property implies that all the morphisms composed (on the either side) with the co-unit morphism c_0, give c_0 again. We shall call this property the *co-unitality* condition:



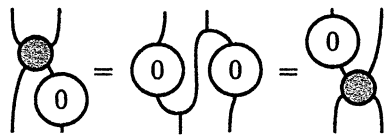
(2.21)

Our second special condition provides a compatibility between the principal morphisms, and the co-unit, by requiring



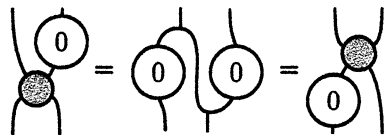
(2.22)

If the above three morphism equalities are fulfilled, then it is easy to see that



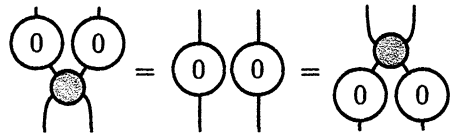
(2.23)

as well as



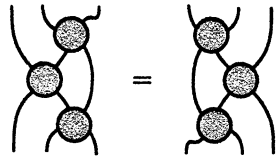
(2.24)

and consequently



(2.25)

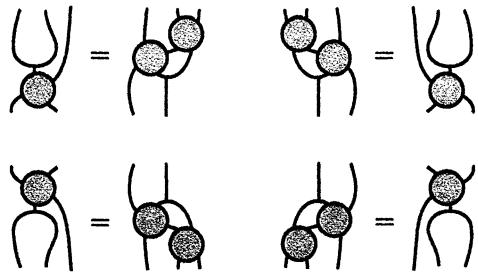
Our third special condition is about the flip over operator. It states that it satisfies the braid equation



(2.26)

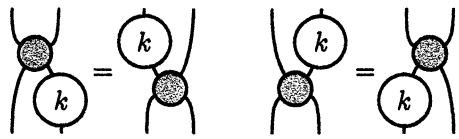
In general, this morphism is not invertible (in particular, not involutive) and it does not satisfy the above braid equation (a simple counterexample is considered in Section 4).

Our fourth special condition deals with relations between the flip-over morphism, and two principal morphisms. It is actually a union of four diagrams:



(2.27)

We shall call it the *complete transportability* condition. If the complete transportability holds, then all the morphisms of our category are transportable, in the most natural way. In particular, we have

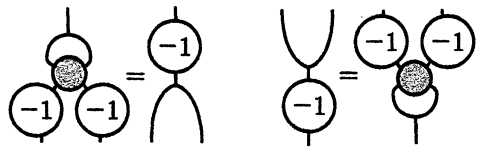


(2.28)

for every $k \in \mathbb{Z}$, the transportability of the circular morphisms (trivial for $k = 1$!). And considering the transportability property to the flip-over morphism itself, we see that it is equivalent to the braid equation.

It is easy to verify that in this scenario, the higher product and co-product structures on all the objects $n \geq 2$, are co/associative. All the objects $n \geq 2$ are naturally equipped with the symmetries given by the natural representation of the braid groups B_n, assuming in addition that the flip-over morphism is *invertible*. In general this is not going to be the case, so the invertibility is a non-trivial condition.

The multiplicativity properties (2.22), are then a logical consequence of transportability and invertibility. We can also prove that c_{-1} is anti-co-multiplicative, in the sense of



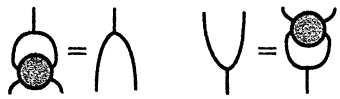
(2.29)

If the transportability condition holds, then all the morphisms of our category have a simple description. An arbitrary morphism $\Psi\colon m \to n$ is given by

$$\Psi = \Phi \circ \Sigma \circ M \tag{2.30}$$

where $\Phi\colon m \to k$ is a direct product of (possibly higher) coproducts, $\Sigma\colon k \to k$ is a morphism from the braid group B_k, arbitrarily coloured by antipodal morphisms c_{-1} and $M\colon k \to n$ is a direct product of (possibly higher) products.

Our fifth special condition is a kind of generalized commutativity and its dual condition (the co-commutativity). It proclaims that

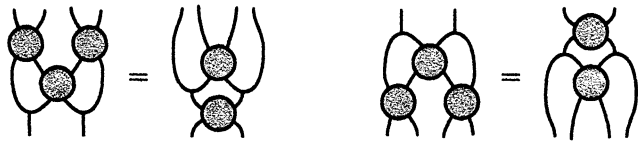


(2.31)

These are actually two logically independent conditions, mutually in reflection. For our C*-algebraic formulation, the second formula is the most

interesting one. As it will become transparent, it *always* holds (somewhat counterintuitive, perhaps).

Our sixth special condition is that the product (or coproduct) morphism intertwines the product (or coproduct) structures on numbers 2 and One. Thus, the intertwining property is

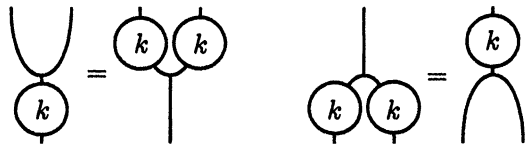


(2.32)

in its mutually dual representations.

Now, if we assume that the commutativity property (2.31) holds, together with the transportability and invertibility, then it is easy to prove that every morphism $\Psi\colon m \to n$ in our category intertwines the corresponding product structures on the objects m and n. This follows from the fact that primary morphisms generate the entire category. As already mentioned, the coproduct intertwines the structures on numbers One and 2, and the commutativity property is precisely the intertwining condition for the diagonal morphism. On the other hand, the composition of intertwining morphisms is clearly again an intertwining morphism. We can also enlarge a given intertwining morphism, by taking left and right direct products with the identity on One. Thanks to the transportability condition, such enlarged morphisms preserve the intertwining property.

So this (and its dual) special scenarios invite us to consider

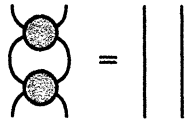


(2.33)

and we conclude that the first of the above equalities holds for all $k \in \mathbb{Z}$, in the specific context diagrammatically reflecting the C*-algebraic framework. The second would correspond to the coproduct intertwining condition.

In particular, by combining this with the convolution property, and applying elementary transformations, we conclude that the composition identity (2.20) holds.

Another important corollary is that the flip-over operator is necessarily involutive, that is



(2.34)

In particular, the symmetries of the braid groups B_n on n are reduced to the symmetries of the permutation groups S_n.

The decomposition of morphisms (2.30) further simplifies, as in this case the middle factor Σ is a permutation, and due to the involutivity of c_{-1} on each line of Σ there can be at most one antipodal morphism. In particular, it follows that all morphisms of the initial object into itself, are exhausted by the circular morphisms.

Remark 2.2. The ideas presented extend beyond the group structure as such. Extended diagrammatic geometrical structures can be introduced in the spirit of [11], where operads of graphs are used, and following the same conceptual framework of the collectivity structures on quantum spaces.

3 A Non-commutative Realization

Let us now consider the question of representing the diagrammatic morphisms in an appropriate category of non-commutative *-algebras. We shall imagine that the given algebraic category somehow describes a universe of quantum spaces. In the spirit of our general philosophy, the realization of diagrammatic partiture correspond to groups in this quantum universe.

Such a quantum group concept, is *very different* from the theory arising from the Hopf algebras. In spite of the fact that it bears certain formal resemblance.

The difference lies in the fact that *all morphisms* in our realization are *-homomorphisms. Compare this with the Hopf algebras, where the antipode is antimultiplicative, in general non-involutive, and exhibits a more complicated relation with the *-structure. Moreover, in the standard formulation there exist no geometrically acceptable counterpart to the diagonal map. On

the other hand, there is a formal similarity. As explained in the previous section, the primary morphisms are building blocks of the diagrammatical co-unit and the antipode, and all this is easily realized with the standard Hopf algebras. However, the realization is not a *geometric realization*, built from the quantum spaces. It is a valid formal realization by vector spaces, their linear maps, and tensor products.

So let us think of C*-algebras as representing quantum spaces, in the usual, dual way, inspired by the classical Gelfand-Naimark theory. However, we shall assume that in addition, the geometrical structure of a quantum space is specified by defining how the space behaves 'collectively' within itself. If A is a C*-algebra representing a quantum space, say X, then we shall consider a family of C*-ideals

$$J_n \subset \overbrace{A \star \cdots \star A}^{n} = A^{\star n}$$

of the n-fold free product algebras. We shall assume that $J_1 = \{0\}$ so we are really interested in $n \geq 2$. On the n-fold product algebras, we can introduce a natural action of product morphisms, embedding morphisms, and also natural actions of the permutation groups S_n on $A^{\star n}$.

We shall further assume that our ideals are invariant, in a natural way, with these three types of morphisms.

Remark 3.1. In more detail, there are n different diagonal morphisms from $A^{\star(n+1)}$ to $A^{\star n}$. We require that all of them map J_{n+1} into J_n. Natural embedding maps $A^{\star n} \rightarrowtail A^{\star(n+1)}$ transform J_n into J_{n+1}. And the permutations acting in $A^{\star n}$ must preserve J_n. It is sufficient to assume this property for elementary transpositions.

Definition 3.2. A collectivity structure on a C*-algebra A (a quantum space X) is the family of ideals J_n satisfying the above mentioned properties. A trivial (or free) collectivity structure is given by $J_n = \{0\}$ for $n \geq 2$.

Now we can easily define the category of quantum spaces, proceeding in the spirit of [14], and requiring all morphisms to be compatible with the corresponding ideals, in the most natural way.

We can also define higher collective powers of A, by factorizing

$$A_n = A^{\star n}/J_n$$

over the ideals J_n, for all $n \in \mathbb{N}$.

The quantum group structure is given by specifying a coproduct morphism $\phi\colon A \to A_2$, which is coassociative and which allows a full representation of the diagrammatic category (and which is compatible with the collectivity ideals J_n). The product (the diagonal map) and the coproduct ϕ being representatives of the primary morphisms, with numbers transforming as $n \mapsto A_n$.

Definition 3.3. A group structure on a quantum space X (C*-algebra A equipped with a collectivity structure) is a unital *-homomorphism $\phi\colon A \to A_2$ satisfying the above properties. It is also natural to say than that X, or (A, ϕ), is a quantum group.

Remark 3.4. A couple of words regarding our notation. A nature of our discourse is such that we deal with individual objects (say things related with the C*-algebra A in our principal context), and their copies within the collectivity structures (say algebras A_n). If X is an individual object, we shall commonly write X_α, $[X]_\alpha$ or $X_{[\alpha]}$ to denote one and the same thing–its copy labeled by α in the corresponding collectivity structure.

4 Examples

4.1 Free Groups and Related Structures

Let us consider an abstract generator x, and let X be the free group generated by x. Clearly, X is isomorphic to the additive group of integers, the isomorphism being given by $X \ni x^k \leftrightarrow k \in \mathbb{Z}$. For $n \geq 2$, the n-fold power of X is defined as the free group $\mathcal{F}_n$, with generators $x_1, \ldots, x_n$, and we can also write $\mathcal{F}_1 = X$. The group $\mathcal{F}_n$ is the free n-fold power of the group X. The principal coproduct $X \to \mathcal{F}_2$ and product $\mathcal{F}_2 \to X$ morphisms are given by

$$x \mapsto x_1 x_2 \qquad\qquad x_1, x_2 \mapsto x \tag{4.1}$$

respectively. Their 'embedded' versions $\phi_j\colon \mathcal{F}_n \to \mathcal{F}_{n+1}$ and $m_j\colon \mathcal{F}_{n+1} \to \mathcal{F}_n$ where $1 \leq j \leq n$ are group homomorphisms specified by

$$\phi_j(x_k) = \begin{cases} x_k & k < j \\ x_j x_{j+1} & k = j \\ x_{k+1} & k > j \end{cases} \qquad m_j(x_k) = \begin{cases} x_k & k \leq j \\ x_{k-1} & k > j. \end{cases} \tag{4.2}$$

It is easy to see that in such a way we obtain the complete realization of the diagrammatic category. The circular morphisms $c_n\colon X \to X$ where $n \in \mathbb{Z}$ are given by $c_n(x) = x^n$, and of course the neutral element and the inverse are here: $\epsilon = c_0$ and $\kappa = c_{-1}$. The canonical twist morphism $\sigma\colon \mathcal{F}_2 \to \mathcal{F}_2$ is given by transposing indexes 1 and 2. Indeed, the outlined steps of the twist diagrammatic process are

$$x_1 \rightsquigarrow x_1 x_2 \rightsquigarrow x_1^{-1} x_2 x_3 \rightsquigarrow x_1 x_1^{-1} x_2 = x_2$$
$$x_2 \rightsquigarrow x_3 x_4 \rightsquigarrow x_2 x_3 x_4^{-1} \rightsquigarrow x_1 x_2^{-1} x_2 = x_1.$$

We shall now explain what are all possible morphisms coming from the diagrams of our diagrammatic category.

Lemma 4.1. *Every group homomorphism $F\colon X \to \mathcal{F}_n$ comes from a diagrammatic morphism.*

Proof. At first, all morphisms between X and X are given by circular morphisms c_k where $k \in \mathbb{Z}$. Let us consider non-trivial morphisms between X and $\mathcal{F}_n$ where $n \geq 2$. Every such a morphism is specified by assigning a given non-trivial element of $\mathcal{F}_n$, to $x \in X$, in other words

$$x \mapsto x_{i_1}^{k_1} \cdots x_{i_m}^{k_m}$$

where $i_1, \ldots, i_m \in \{1, \ldots, n\}$ are such that no neighbours (if there are!) in this index sequence coincide, and $k_1, \ldots, k_m$ are all non-zero integers. We shall now reproduce this expression by using the product, coproduct, inverse and transposition morphisms. At first, we send x to a sufficiently large sequence of generators of $\mathcal{F}_N$, by successive application of the coproduct:

$$x \mapsto x_1 \cdots x_N = y \qquad N = |k_1| + \cdots + |k_m|.$$

The sequence $1, \ldots, N$ naturally maps to indexes $i_1, \ldots, i_m$: We start with i_1 repeating $|k_1|$ times, and end with i_m repeating $|k_m|$ times. If the exponent k_λ is negative, on the corresponding sequence of generators we apply the inverse morphism κ. Our next step is to permute the set $1, \ldots, N$ such that, after the reparametrization, the inverse image of every index i_λ consists of consecutive numbers. The permutation can be carried out by composing the transposition morphisms, applied on the product y. We next apply the product morphisms, to objects corresponding to each such an inverse image. By doing this, we end up in $\mathcal{F}_d$ where d is the number of different indexes in the sequence $i_1, \ldots, i_m$. In other words,

$$x \mapsto x_1 \cdots x_N \mapsto x_{j_1}^{k_1} \cdots x_{j_m}^{k_m}$$

where $j_1, \ldots, j_m \in \{1, \ldots, d\}$ is a sequence equivalent (it mirrors repetitions) to $i_1, \ldots, i_m$. The final step is to insert $\mathcal{F}_d$ into $\mathcal{F}_n$, and via an appropriate permutation represented by transposition morphisms, transform $j_1, \ldots, j_m$ into $i_1, \ldots, i_m$. □

Proposition 4.2. *Every group homomorphism $F\colon \mathcal{F}_p \to \mathcal{F}_n$ is a representation of a diagrammatic morphism.*

Proof. For $p = 1$ the statement is that of the previous lemma. Let us therefore assume that $p \geq 2$. The group morphism F is defined by arbitrarily specifying $F(x_1), \ldots, F(x_p)$. Each of these specifications has its own morphism $F_1, \ldots, F_p$ from X to $\mathcal{F}_n$, which according to the above lemma is diagrammatically representable. By taking the direct product of them we obtain a morphism from $\mathcal{F}_p$ to $\mathcal{F}_{pn}$. Finally, by 'glueing' the strings corresponding to the indexes having the same modulus n (and this is achieved by a shuffling permutation, followed by the corresponding p-fold products) we obtain the desired morphism $F\colon \mathcal{F}_p \to \mathcal{F}_n$. □

This construction works for free groups. Let us now consider what happens when we factorize X over cyclic relations $x^n = 1$, obtaining the cyclic groups $\mathbb{Z}_n$, where $n \geq 2$. This is the only possible type of a relation here. We would like to have all diagrammatic morphisms representable in this scenario. This requires introducing additional relations in the direct product objects.

Let us first consider the simplest case $n = 2$. Since x is involutive, all the generators of higher order products are involutive too. The coproduct acts as $x \mapsto x_1 x_2$, so $x_1 x_2$ is involutive. However this implies that x_1 and x_2 commute, and similarly all generators must commute. Therefore the minimal solution for the product objects is the classical solution, the n-fold product is just the classical Cartesian product $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

The next simplest case is $n = 3$. However, here the situation changes drastically. Since all diagrammatical morphisms must be represented, and since all elements of $\mathcal{F}_n$ can be obtained by transforming the generator x, we conclude that all elements y in the higher-order products must satisfy $y^3 = 1$. The same reasoning applies for any number $n \geq 3$.

So the simplest algebra, at the level of d particles, is a group algebra of the group generated by d symbols and a universal cyclicity condition: the n-th power of every element must be trivial. The *universal Burnside groups* $B(d, n)$ are precisely defined in this way. We factorize the free group $\mathcal{F}_d$ over the normal subgroup generated by all possible n-th powers.

The groups $B(d, n)$ can be finite or infinite. It is well known that for $n = 3, 4, 6$ all Burnside groups are finite. If n is sufficiently large however, the groups $B(d, n)$ will be infinite. In such an infinite case, there always exists a *universal finite group* $B^*(d, n)$ such that every finite group with d generators in which n-th powers vanish, is a homomorphic image of $B^*(d, n)$ in a natural way.

Remark 4.1. An in-depth exposition of the classical Burnside problem and related problematics, from algebraic and geometric perspectives, can be found in books [8] and [10]. A brief overview can be found in [9], which also contains a lot of interesting examples of discrete groups, and general considerations with graphs and trees.

So going back to the simplest really non-trivial case $d = 2$ and $n = 3$: We have two generators and the cube of everything vanishes. In Appendix A we listed all the elements of this group, there are 27 of them. In general, the order of the group $B(d, 3)$ is $3^{q(d)}$ where

$$q(d) = d + \binom{d}{2} + \binom{d}{3}.$$

On the other hand, the order of the group $B(2,4)$ is $2^{12} = 4096$, while the group $B(2,6)$ is of order 65229815808. More generally, the groups $B(d,6)$ have exactly $2^{a(d)}3^{b(d)}$ elements, where

$$a(d) = 1 + (d-1)3^{q(d)} \qquad b(d) = 1 + (d-1)2^d.$$

The group algebras of all these groups $B(d,n)$ provide a universal collectivity structure for the cyclic groups $\mathbb{Z}_n$. In the infinite case, we can also consider a reduced collectivity structure based on the finite groups $B^*(d,n)$. In both cases, as it follows from the general theory, the corresponding group algebras are naturally interpretable as geometric quantum groups themselves.

Remark 4.2. We can naturally extend these considerations to include unitary $m \times m$ matrices, instead of using single unitary generators. In such a way we arrive to matrix C*-algebras $A(d,n,u)$. By definition, they are generated by d independent copies of the abstract unitary $m \times m$ matrix u, such that the n-th power of every product combination of matrices and their inverses, is trivial. Every $A(d,n,u)$ defines the collectivity structure of order d, for the initial algebra $A(n,u)$ which is defined by a system of relations equivalent to a single matrix relation $u^n = 1$.

By construction, there is always a natural group structure, the coproduct is represented by the standard matrix comultiplication, as in the example of universal unitary quantum groups, discussed below.

In the special case $n = 2$, the collectivity relations are that of mutual commutation of all copies of u, which must be involutive.

4.2 Set Theoretic Realizations

Let us consider diagrammatic realizations with classical sets and their maps. The initial object One is represented by a (nonempty, let us assume) set Λ. We shall assume that the higher order objects are represented by sets

$$\Lambda^n \subseteq \overbrace{\Lambda \times \cdots \times \Lambda}^{n},$$

subsets of the n-fold direct product of Λ with itself. The simplest realization is to take Λ^n to be the full n-fold direct products. This leads exactly to a classical group structure on Λ, as it will become clear in a moment.

In general, the diagonal map $\delta\colon \Lambda \to \Lambda^2$ is of the form

$$\delta(x) = (l(x), r(y)) \tag{4.3}$$

where $l, r\colon \Lambda \to \Lambda$ are maps satisfying

$$l^2 = l \qquad r^2 = r \qquad lr = rl. \tag{4.4}$$

The above is equivalent to the coassociativity of δ. In other words, l and r are mutually commuting projectors on their respective images. The binary product map $\bullet\colon \Lambda^2 \to \Lambda$ is associative, and must satisfy the invertibility conditions (2.3). In other words the maps

$$(x, y) \rightsquigarrow (l(x), r(x) \bullet y) \qquad (x, y) \rightsquigarrow (x \bullet l(y), r(y))$$

are bijective on Λ^2.

Remark 4.3. This implies that Λ^2 must be contained within $\mathrm{im}(l) \times \mathrm{im}(r)$. If we in addition ask that both projections of Λ^2 on Λ be surjective (which trivially will be satisfied if $\Lambda^2 = \Lambda \times \Lambda$) then both l and r trivialize to the identity map on Λ and we have $\delta(x) = (x, x)$, the standard diagonal map. In this case $\bullet$ begins to resemble a classical group structure on Λ, and as we have just mentioned if all Λ^n are full powers of Λ the theory is equivalent to the classical groups.

Here is a simple non-standard realization. Let G be a group, and X a set equipped with a right action of G. Let us define $\Lambda = X \sqcup G$, and

$$\Lambda^n = \Lambda \times \overbrace{G \times \cdots \times G}^{n-1} \tag{4.5}$$

for $n \geq 2$. In particular $\Lambda^2 = \Lambda \times G = (X \times G) \sqcup (G \times G)$. The product $\bullet\colon \Lambda^2 \to \Lambda$ is given by combining the group product and the right action map. The left and right projectors are defined as

$$l(x) = x \qquad l(g) = g \qquad r(g) = g \qquad r(x) \in G \tag{4.6}$$

for $x \in X$ and $g \in G$. So the restriction $r\colon X \to G$ is arbitrary. It is easy to verify that the bijectivity property (2.3) holds. The co-unit morphism is given by

$$\epsilon(x) = xr(x)^{-1} \qquad \epsilon(g) = e \tag{4.7}$$

where $e \in G$ is the neutral element. The antipodal morphism is given by

$$\kappa(x) = xr(x)^{-2} \qquad \kappa(g) = g^{-1} \tag{4.8}$$

and in general we have

$$c_n(x) = xr(x)^{n-1} \qquad c_n(g) = g^n \tag{4.9}$$

for every $n \in \mathbb{Z}$. The flip-over morphism $\sigma\colon \Lambda^2 \to \Lambda^2$ is given by

$$\sigma(x,g) = (\epsilon(x)g, r(x)) \qquad \sigma(h,g) = (g,h) \tag{4.10}$$

as it follows directly from the definition of σ and the above expressions for the co-unit and the antipode.

We see that σ in general is not bijective. It also does not satisfy the braid identity, in general.

Proposition 4.3. *The flip-over morphism satisfies the braid identity if and only if $r\colon X \to G$ is covariant the the sense that*

$$r(xg) = r(x)g \tag{4.11}$$

for each $x \in X$ and $g \in G$. In particular, we see that the action of G is necessarily free in this case. The same condition is equivalent to σ being involutive.

Proof. We have to check the braid equation on $X \times G \times G$. Calculating both sides of the braid equation gives

$$(x,g,h) \rightsquigarrow (x,h,g) \rightsquigarrow (xr(x)^{-1}h, r(x), g) \rightsquigarrow (xr(x)^{-1}h, g, r(x))$$

as well as

$$\begin{aligned}(x,g,h) \rightsquigarrow (xr(x)^{-1}g, r(x), h) \rightsquigarrow (xr(x)^{-1}g, h, r(x)) \rightsquigarrow \\ \rightsquigarrow (xr(x)^{-1}gr(xr(x)^{-1}g)^{-1}h, r(xr(x)^{-1}g), r(x))\end{aligned}$$

and comparing the final results, we deduce that the braid equation is equivalent to the covariance condition. The square of the flip-over operator is

$$\sigma^2(x,g) = \sigma(xr(x)^{-1}g, r(x)) = (xr(x)^{-1}gr(xr(x)^{-1}g)^{-1}r(x), r(xr(x)^{-1}g))$$

and similarly comparing the result with (x,g) we deduce that the involutivity is also equivalent to the covariance condition. On the other hand, the covariance condition implies that the action is free, indeed if $xg = x$ then $r(xg) = r(x)g = r(x)$ and it follows that g must be trivial. □

Let us analyse the transportability properties of the primary morphisms. It turns out that the left transportability of both morphisms is fulfilled automatically. Indeed,

$$(x,g,h) \rightsquigarrow (xr(x)^{-1}g, r(x), h) \rightsquigarrow (xr(x)^{-1}g, h, r(x)) \rightsquigarrow (xr(x)^{-1}gh, r(x))$$

which gives the same result as $(x,g,h) \rightsquigarrow (x,gh) \rightsquigarrow (xr(x)^{-1}gh, r(x))$. Furthermore,

$$(x,g) \rightsquigarrow (x, r(x), g) \rightsquigarrow (x, g, r(x)) \rightsquigarrow (xr(x)^{-1}g, r(x), r(x))$$

which gives the same as $(x,g) \rightsquigarrow (xr(x)^{-1}g, r(x)) \rightsquigarrow (xr(x)^{-1}g, r(x), r(x))$, so both morphisms are left transportable.

Proposition 4.4. *The property of right transportability for either of the primary morphisms, is equivalent to the covariance condition.*

Proof. Verifying the right transportability for the composition. We compute

$$(x,g,h) \rightsquigarrow (x,h,g) \rightsquigarrow (xr(x)^{-1}h, r(x), g) \rightsquigarrow (xr(x)^{-1}h, r(x)g)$$

and on the other hand $(x,g,h) \rightsquigarrow (xg,h) \rightsquigarrow (xgr(xg)^{-1}h, r(xg))$. Checking the right transportability of the diagonal map gives

$$(x,g) \rightsquigarrow (x,g,g) \rightsquigarrow (xr(x)^{-1}g, r(x), g) \rightsquigarrow (xr(x)^{-1}g, g, r(x))$$

and on the other hand

$$(x,g) \rightsquigarrow (xr(x)^{-1}g, r(x)) \rightsquigarrow (xr(x)^{-1}g, r(xr(x)^{-1}g), r(x)).$$

We see that both conditions are equivalent to the covariance property. □

Remark 4.4. It is natural to consider a generalization of the surjectivity property, mentioned in the previous remark. To postulate that all coordinate projection maps from Λ^{n+1} to Λ^n be surjective. As already explained, since this includes the surjectivity at the level $n = 2$, it implies that δ is the standard diagonal map. The example above clearly violates this property. Its main conceptual value is that it shows that the braid equation (as well as involutivity and functoriality) is not a consequence of the diagrammatic axioms. If we think of Λ^n as some geometrical collectivity on Λ, then the regularity comes as a natural postulate.

One interesting realization of the above scheme, related to the free groups and Burnside groups considered at the beginning of this section, is to take a classical group G, and fix a number $n \geq 2$. Then define the set Λ to be all elements of G whose n-th power vanishes (the order of which divides n). Then restrict to the subspaces $\Lambda^d \subseteq \Lambda \times \cdots \times \Lambda$ of the d-fold direct products, consisting exactly of such multiplets from Λ generating a subgroup of G contained in Λ.

As another concrete non-classical realization of the above scheme, let us mention the (non-associative) group of octonions. The octonionic algebra $\mathbb{O}$ possesses the following important *di-associativity property*: Every two octonions $x, y \in \mathbb{O}$ generate an associative subalgebra $o_{x,y}$ of $\mathbb{O}$. In the generic case, this algebra will be isomorphic to the noncommutative cuaternions field $\mathbb{H}$.

We define $\Lambda = \Lambda^1 = S^7$, the set of unitary octonions. Furthermore for every $n \geq 2$ we set

$$\Lambda^n = \Big\{(o_1, \ldots, o_n) \in \Lambda \;\Big|\; \mathrm{gen}(o_1, \ldots, o_n) \text{ is associative}\Big\}. \tag{4.12}$$

By construction, we have $\Lambda^2 = \Lambda \times \Lambda$. On the other hand, of course, for $n \geq 3$ the spaces Λ^n are strictly contained in the n-fold products $\Lambda \times \cdots \times \Lambda$.

The diagrammatic realization is then given by the product in Λ and the diagonal map, together with the assignements $n \mapsto \Lambda^n$.

Remark 4.5. This collectivity structure can be further extended to the full n-fold direct products of Λ, by using the appropriate exotic extensions of the diagrammatic category representation, as explained in [7].

4.3 Unitary Matrix Quantum Groups and Friends

Let us consider an integer $m \geq 2$ and abstract generators u_{ij} forming a symbolic $m \times m$ matrix u. Let $\mathcal{A}$ be a *-algebra generated by symbols u_{ij} and the relations

$$\sum_{k=1}^{m} u_{ik} u_{jk}^* = \delta_{ij} = \sum_{k=1}^{m} u_{ki}^* u_{kj} \tag{4.13}$$

in other words, we are postulating the unitarity of the matrix u. Let A be the C*-algebraic completion of $\mathcal{A}$.

Let us postulate that all the ideals defining the collectivity structure are trivial, in other words

$$J_n = \{0\} \qquad A_n = \overbrace{A \star \cdots \star A}^{n}.$$

The product maps (basic and the higher-level) are the standard products in the collection $(A_n)_{n \in \mathbb{N}}$. The coproduct map $\phi \colon A \to A \star A$ is given by

$$\phi(u_{ij}) = \sum_{k=1}^{m} [u_{ik}]_1 [u_{kj}]_2 \tag{4.14}$$

on the generators, and extended to the whole of the algebra A as a unital *-homomorphism. In symbolic matrix notation, this means

$$\phi(u) = [u]_1 [u]_2$$

and we see that if u is unitary then $\phi(u)$ is unitary as well. Therefore the relations defining A and the definition of ϕ are compatible, and the extension exists (and is unique).

In such a way we obtain a quantum group $\mathrm{QU}(m)$. All diagrammatic morphisms are represented by *-homomorphisms between the algebras A_n. In particular, the representations $\epsilon \colon A \to A$ and $\kappa \colon A \to A$ of the co-unit morphism and the antipode morphism are given by

$$\epsilon(u_{ij}) = \delta_{ij} \qquad \kappa(u_{ij}) = u_{ji}^*.$$

More generally, it is easy to see that the circular morphisms c_n are represented by unital *-homomorphisms of A acting as $u \mapsto u^n$, for all $n \in \mathbb{Z}$.

The classical part of QU(m) consists precisely of classical unitary matrices. If we factorize A_n over their commutants, we obtain the classical unitary group U(m). In this sense U(m) is a subgroup of QU(m).

The flip-over morphism is represented by the canonical twist in $A_2 = A \star A$, the *-automorphism which exchanges the first and the second position. It is obviously involutive (as it should be, according to the general theory) and satisfies the braid equation. In other words we have the permutation symmetry.

Essentially the same considerations apply in the case of the presence of *additional relations* between the generators u_{ij}, at both the individual and collectivity levels. These examples introduce an interesting subtlety in the game: The relations must be 'preserved' by all diagrammatic morphisms. And since all these morphisms are generated by the coproduct and the product/diagonal map, it follows that we only have to check the compatibility of the ideal of relations, with these two basic morphisms, and the antipodal map–reflecting the invertibility of the two basic combinations of the primordial morphisms.

If we symbolically denote by $r(u) = 0$ the algebraic relations for a matrix u defining the individual algebra A, then in higher-order algebras we must have

$$r\left([u^{n_1}]_{i_1} \cdots [u^{n_k}]_{i_k}\right) = 0$$

because the matrices of the form $[u^{n_1}]_{i_1} \cdots [u^{n_k}]_{i_k}$ are the result of the action by all possible diagrammatic morphisms on the matrix u. In particular, at the individual level, the relations should be stable under replacing u by any integer power u^n.

As a concrete example, let us consider at the individual level, relations

$$u_{ij}u_{kl} = u_{kl}u_{ij} \qquad u_{ij}u_{kl}^* = u_{kl}^*u_{ij}. \tag{4.15}$$

In other words, the entries of u generate a commutative C*-algebra A. It is clear that all powers of u share this commutativity property. So we have $A = C[\mathrm{U}(m)]$. On higher-order levels, we have to postulate that any product of (powers of) copies of u is a commutative matrix, in this sense. In spite of this commutativity, the algebras A_n for $n \geq 2$ are all

non-commutative. Classical individuality with quantum collectivity. The phenomenon is already present in the simplest case $m = 1$, of Circle.

Definition 4.6. Every structure based on a unitary matrix u as explained above, is called a *unitary matrix quantum group.*

Let us consider a more general setting, in which there is an 'auxiliary' unitary matrix Q, which together with the mutually identical matrices $([u]_k)_{k\in\mathbb{N}}$ generates all the collectivity algebras, and such that the morphisms are *linear*, with respect to the entries of Q. Let us assume that formula (4.14) for the coproduct is replaced by

$$\phi(u_{ij}) = \sum_{kl=1}^{m} [u_{ik}]_1 Q^*_{lk} [u_{lj}]_2 \qquad u \mapsto [u]_1 Q^* [u]_2. \tag{4.16}$$

It is clear that this definition (obviously generalizing the initial matrix coproduct where Q is the identity $m \times m$ matrix) implies the coassociativity property of the morphism. The coproduct and the co-unit morphisms are given by:

$$\epsilon(u_{ij}) = Q_{ij} \qquad \kappa(u_{ij}) = \sum_{kl=1}^{m} Q_{ik} u^*_{lk} Q_{lj} \qquad u \mapsto Q \mid Qu^*Q. \tag{4.17}$$

Remark 4.7. We can intuitively interpret the quantum space, corresponding to the algebra generated by the entries of Q, as the playground for the co-unit fluctuations, where the quantum space represents all concrete 'possibilities' for the co-unit: the points of this, possibly 'pointless' space. This algebra is the image of the co-unit morphism. As we shall see below, very interesting quantum spaces, without any classical points, can be in such a way endowed with a group structure. To mention a concrete example: quantum torus and quantum Euclidean planes.

A very important special case, in a sense complementary to the initial definition of the matrix quantum unitary group (where Q is trivial), arises when transforms isomorphically the C*-algebra of u on that of Q. This, for classical groups, corresponds to allowing any point of G to play the

role of the neutral element—by considering the covariant ternary product $G \times G \times G \to G$ given by

$$(\alpha, \beta, \gamma) \mapsto \alpha\beta^{-1}\gamma$$

as explained in detail in [6]. Then the coordinate β is interpretable as this 'fluctuating dimension'. If we keep β fixed, then the ternary product reduces to a binary product between α and γ, producing the shifted group structure, so that β is the neutral element, and the shifted inverse is given by $x \mapsto \beta x^{-1}\beta$. All these products give isomorphic group structures (connectable by left/right translations) and generate one and the same ternary product. In the quantum case in general, the coordinate β, does not exist in the form of a 'single' point.

We shall now explain how it is possible to naturally go back and forth between quantum groups with fluctuating co-unit, and the standard structures where Q is trivial.

Let us consider a (u, Q) based quantum group G and define a unitary $m \times m$ matrix

$$v = uQ^*. \tag{4.18}$$

The coproduct formula (4.16) gives

$$\phi(v_{ij}) = \sum_{k=1}^{m} [v_{ik}]_1 [v_{kj}]_2 \tag{4.19}$$

in other words, it is the standard matrix coproduct. Not surprisingly, we also have $\epsilon(v_{ij}) = \delta_{ij}$ and $\kappa(v_{ij}) = v_{ji}^*$.

Proposition 4.5. *Thus, if we consider the C*-subalgebra B of A generated by the entries of v, together with the corresponding collectivity subalgebras B_n, the entire system of diagrammatic morphisms restricts on them, and we obtain a standard matrix quantum group based on v.* □

Conversely, let us assume that a standard quantum matrix group is given, based on a unitary $m \times m$ matrix v and a system of C*-algebras B_n. Let us consider a C*-algebra C generated by the entries of nanother unitary

$m \times m$ matrix Q, and let us define A as a product $\boxtimes$ between C and B, and also extend this definition to higher-order algebras

$$A_n = B_n \boxtimes C.$$

Let us assume that the diagrammatic morphisms are extendible, from B_n to A_n, by postulating their C-linearity. And define a unitary $m \times m$ matrix

$$u = vQ.$$

It is a matter of straightforward verification, that equations (4.16) and (4.17) hold.

Proposition 4.6. *Thus, the system of C*-algebras A_n together with the extended diagrammatic morphisms, constitutes a matrix quantum group based on (u, Q).* □

Remark 4.8. In particular, in the definition of algebras A_n, we could have used the standard C*-algebraic tensor product, or the free one as $\boxtimes$. However, in resonance with our geometrical interpretation, it is also natural to assume that the assignement $Q \rightsquigarrow vQ = u$ extends to a *-homomorphism (necessarily iso) between C and the 'active' individual algebra generated by the entries of u. The passage from the standard quantum group to a fluctuating co-unit one, is highly non-unique, in contrast to the well defined association $u \mapsto v$. It is a very interesting problematics per se, to analyze possible forms of non-trivial entaglements between this co-unit fluctuations space, and the initial quantum space.

Remark 4.9. It is also worth observing that we can extend this construction beyond matrix structures as such, by considering a coalgebra $\phi\colon \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ which is a generating space of the *-algebra $\mathcal{A}$ over which we model the quantum group. Such a setup can include a 'rudimentary' counit $\epsilon\colon \mathcal{L} \to \mathbb{C}$ and the antipode $\kappa\colon \mathcal{L} \to \mathcal{A}$ so that we have the basic equations

$$\epsilon(q) = q^{(1)}\kappa(q^{(2)}) = \kappa(q^{(1)})q^{(2)}.$$

Of course, the algebra $\mathcal{A}$ could already be equipped with a standard Hopf algebra structure, which when restricted to $\mathcal{L}$ gives the above mentioned maps. We shall soon meet a couple of concrete examples of this kind.

4.4 Quantum SU(2) Groups

In the spirit of the cross product construction of Appendix C, we shall present a non-universal quantum version of the unitary U(2) group. As already discussed, the unitarity condition as such, naturally leads to 'truly quantum' groups QU(n) for each $n \geq 1$. The groups that we are going to construct are all interpretable as deformation-style subgroups of QU(2). In a certain sense, they are a natural modification of the quantum SU(2) group of [16].

We start by considering the classical SU(2) group and its subgroup H consisting of diagonal matrices. So our initial Hopf algebra will be generated by a single unitary denoted by $\sqrt{\mu}$ and the matrices M and U which are given by

$$M = \begin{pmatrix} \mu^{1/2} & 0 \\ 0 & \bar{\mu}^{-1/2} \end{pmatrix} \qquad U = \begin{pmatrix} \alpha_0 & -\gamma_0^* \\ \gamma_0 & \alpha_0^* \end{pmatrix} \tag{4.20}$$

where $|\alpha_0|^2 + |\gamma_0|^2 = 1$. It is easy to see that

$$MUM^* = \begin{pmatrix} \alpha_0 & -\mu\gamma_0^* \\ \bar{\mu}\gamma_0 & \alpha_0^* \end{pmatrix}$$

and therefore

$$\sqrt{\Delta}\alpha_0\sqrt{\Delta^*} = \alpha_0 \quad \sqrt{\Delta}\alpha_0^*\sqrt{\Delta^*} = \alpha_0^* \quad \sqrt{\Delta}\gamma_0\sqrt{\Delta^*} = \bar{\mu}\gamma_0 \quad \sqrt{\Delta}\gamma_0^*\sqrt{\Delta^*} = \mu\gamma_0^*,$$

where we used the symbol $\sqrt{\Delta}$ for the elementary automorphism generator.

The matrix u generating the quantum group G and playing the role of the fundamental representation, is given by

$$u = M^{-1}\sqrt{\Delta}U = \begin{pmatrix} \bar{\mu}^{1/2}\sqrt{\Delta}\alpha_0 & -\bar{\mu}^{1/2}\sqrt{\Delta}\gamma_0^* \\ \mu^{1/2}\sqrt{\Delta}\gamma_0 & \mu^{1/2}\sqrt{\Delta}\alpha_0^* \end{pmatrix} = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}$$

where we defined

$$\alpha = \bar{\mu}^{1/2}\sqrt{\Delta}\alpha_0 \qquad \gamma = \mu^{1/2}\sqrt{\Delta}\gamma_0 \tag{4.21}$$

It is easy to see that the following commutation relations hold:

$$\alpha\gamma = \bar{\mu}\gamma\alpha \qquad \alpha^*\gamma^* = \bar{\mu}\gamma^*\alpha^* \tag{4.22}$$

$$\alpha\gamma^* = \mu\gamma^*\alpha \qquad \alpha^*\gamma = \mu\gamma\alpha^* \tag{4.23}$$

$$\alpha\alpha^* = \alpha^*\alpha \qquad \gamma\gamma^* = \gamma^*\gamma \tag{4.24}$$

$$\alpha^*\alpha + \gamma^*\gamma = 1 \tag{4.25}$$

$$\begin{aligned} \sqrt{\Delta}\gamma^* &= \mu\gamma^*\sqrt{\Delta} \quad \sqrt{\Delta}\gamma = \bar{\mu}\gamma\sqrt{\Delta} \\ \sqrt{\Delta}\alpha &= \alpha\sqrt{\Delta} \qquad \sqrt{\Delta}\alpha^* = \alpha^*\sqrt{\Delta} \end{aligned} \tag{4.26}$$

It is worth observing that relations (4.22)–(4.25) are equivalent to the unitarity condition for the matrix

$$v = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}. \tag{4.27}$$

From the above decomposition of u we see that Δ plays the role of a quantum determinant. It is clear that $\alpha = u_{11}$ and $\gamma = u_{21}$, while $\gamma^* = -\bar{\mu}u_{12}\Delta^{-1}$ and $\alpha^* = u_{22}\Delta^{-1}$. We have

$$\Delta = u_{11}u_{22} - \bar{\mu}u_{21}u_{12} = u_{11}u_{22} - \mu u_{12}u_{21}$$

which gives another justification for thinking of Δ as a determinant.

From the definition of our generators, and the construction of the direct products, we see that the relations in the 2-particle algebra are

$$\begin{aligned} \alpha_1\gamma_2 = \bar{\mu}_1\gamma_2\alpha_1 \quad &\leftrightarrow \quad \alpha_1^*\gamma_2^* = \bar{\mu}_1\gamma_2^*\alpha_1^* \\ \alpha_1^*\gamma_2 = \mu_1\gamma_2\alpha_1^* \quad &\leftrightarrow \quad \alpha_1\gamma_2^* = \mu_1\gamma_2^*\alpha_1 \end{aligned} \tag{4.28}$$

$$\begin{aligned} \alpha_2\gamma_1 = \bar{\mu}_2\gamma_1\alpha_2 \quad &\leftrightarrow \quad \alpha_2^*\gamma_1^* = \bar{\mu}_2\gamma_1^*\alpha_2^* \\ \alpha_2^*\gamma_1 = \mu_2\gamma_1\alpha_2^* \quad &\leftrightarrow \quad \alpha_2\gamma_1^* = \mu_2\gamma_1^*\alpha_2 \end{aligned} \tag{4.29}$$

$$\begin{aligned} \gamma_1\gamma_2 = \frac{\mu_2}{\mu_1}\gamma_2\gamma_1 \quad &\leftrightarrow \quad \gamma_1^*\gamma_2^* = \frac{\mu_2}{\mu_1}\gamma_2^*\gamma_1^* \\ \gamma_1\gamma_2^* = \frac{\mu_1}{\mu_2}\gamma_2^*\gamma_1 \quad &\leftrightarrow \quad \gamma_1^*\gamma_2 = \frac{\mu_1}{\mu_2}\gamma_2\gamma_1^* \end{aligned} \tag{4.30}$$

$$\begin{aligned} \alpha_1\alpha_2 = \alpha_2\alpha_1 \quad &\leftrightarrow \quad \alpha_1^*\alpha_2^* = \alpha_2^*\alpha_1^* \\ \alpha_1\alpha_2^* = \alpha_2^*\alpha_1 \quad &\leftrightarrow \quad \alpha_1^*\alpha_2 = \alpha_2\alpha_1^* \end{aligned} \tag{4.31}$$

$$\begin{aligned} \Delta_1\alpha_2\Delta_1^* = \alpha_2 \quad &\leftrightarrow \quad \Delta_1\alpha_2^*\Delta_1^* = \alpha_2^* \\ \Delta_2\alpha_1\Delta_2^* = \alpha_1 \quad &\leftrightarrow \quad \Delta_2\alpha_1^*\Delta_2^* = \alpha_1^* \end{aligned} \tag{4.32}$$

$$\begin{aligned} \Delta_1\gamma_2\Delta_1^* = \bar{\mu}_1^2\gamma_2 \quad &\leftrightarrow \quad \Delta_1\gamma_2^*\Delta_1^* = \mu_1^2\gamma_2^* \\ \Delta_2\gamma_1\Delta_2^* = \bar{\mu}_2^2\gamma_1 \quad &\leftrightarrow \quad \Delta_2\gamma_1^*\Delta_2^* = \mu_2^2\gamma_1^* \end{aligned} \tag{4.33}$$

$$\begin{aligned} \Delta_1\Delta_2 = \Delta_2\Delta_1 &\leftrightarrow \Delta_1^*\Delta_2 = \Delta_2\Delta_1^* \\ \Delta_1\Delta_2^* = \Delta_2^*\Delta_1 &\leftrightarrow \Delta_1^*\Delta_2^* = \Delta_2^*\Delta_1^* \end{aligned} \tag{4.34}$$

and in addition we have that μ_1 and μ_2 commute with everything.

In accordance with the general construction, the coproduct ϕ acts as the matrix multiplication

$$\begin{aligned} &\alpha \mapsto \alpha_1\alpha_2 - \bar{\mu}_1\gamma_1^*\gamma_2\Delta_1 \quad \gamma \mapsto \gamma_1\alpha_2 + \Delta_1\alpha_1^*\gamma_2 \quad \mu \mapsto \mu_1\mu_2 \\ &\alpha^* \mapsto \alpha_2^*\alpha_1^* - \bar{\mu}_2\gamma_1\gamma_2^*\Delta_2 \quad \gamma^* \mapsto \alpha_2^*\gamma_1^* + \gamma_2^*\alpha_1\Delta_1^{-1} \quad \Delta \mapsto \Delta_1\Delta_2 \end{aligned} \tag{4.35}$$

while the co-unit ϵ and the antipode κ are given by

$$\begin{aligned} \epsilon(\alpha) = \epsilon(\alpha^*) = \epsilon(\mu) = \epsilon(\Delta) = 1 \qquad \epsilon(\gamma) = \epsilon(\gamma^*) = 0 \\ \kappa(\Delta) = \Delta^* = \Delta^{-1} \qquad \kappa(\mu) = \bar{\mu} = \mu^{-1} \\ \kappa(\alpha) = \alpha^* \quad \kappa(\alpha^*) = \alpha \quad \kappa(\gamma) = -\mu\gamma\Delta^{-1} \quad \kappa(\gamma^*) = -\mu\gamma^*\Delta. \end{aligned} \tag{4.36}$$

Let $\mathcal{A}$ be a subalgebra of the complete algebra $\mathcal{A}'$ generated by the entries of u. It is clear that $\mathcal{A}$ is strictly smaller than $\mathcal{A}'$, for example μ or $\bar{\mu}$ or any non-constant function of μ is not in $\mathcal{A}$. Actually the following natural isomorphism holds

$$\mathcal{A}' \leftrightarrow \mathrm{Pol}(\mu) \otimes \mathcal{A} \tag{4.37}$$

in other words every element q of $\mathcal{A}'$ can be uniquely written as a finite sum

$$q = \sum_{n\in\mathbb{Z}} \mu^n a_n \qquad a_n \in \mathcal{A}.$$

From the definition of $\mathcal{A}'$ it follows that there exists a unique homomorphism $\epsilon\colon \mathcal{A} \to \mathbb{C}$ defined by

$$\epsilon(\alpha) = \epsilon(\alpha^*) = \epsilon(\Delta) = \epsilon(\Delta^*) = \epsilon(\mu) = \epsilon(\bar{\mu}) = 1 \qquad \epsilon(\gamma) = \epsilon(\gamma^*) = 0$$

which on $\mathcal{A}$ is the same as saying

$$\epsilon(u_{ij}) = \delta_{ij} \qquad i,j \in \{1,2\}.$$

It is clear that the map ϵ is *-preserving. This will be our *counit* map. Let us now consider the adjoint/inverse matrix

$$u^\dagger = u^{-1} = \begin{pmatrix} \alpha^* & \gamma^* \\ -\bar{\mu}\Delta^*\gamma & \Delta^*\alpha \end{pmatrix} = \begin{pmatrix} \alpha^* & \gamma^* \\ -\mu\gamma\Delta^* & \alpha\Delta^* \end{pmatrix}$$

which is also unitary, and of the same form as the initial matrix, if we make the substitutions

$$\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \rightsquigarrow \begin{pmatrix} \alpha^* & \gamma^*\Delta \\ -\mu\gamma\Delta^* & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \Delta^* \end{pmatrix} v^\dagger \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} \qquad \begin{matrix} \mu \rightsquigarrow \bar{\mu} \\ \Delta \rightsquigarrow \Delta^*. \end{matrix}$$

In other words, there exists (a unique) *-homomorphism $\kappa\colon \mathcal{A}' \to \mathcal{A}'$ such that

$$\kappa(\alpha) = \alpha^* \;\leftrightarrow\; \kappa(\alpha^*) = \alpha \quad | \quad \kappa(\gamma) = -\mu\gamma\Delta^* \;\leftrightarrow\; \kappa(\gamma^*) = -\bar{\mu}\gamma^*\Delta$$
$$\kappa(\mu) = \bar{\mu} \qquad \kappa(\Delta) = \Delta^*$$

and in particular we see that κ is involutive, in other words

$$\kappa^2(a) = a$$

for every $a \in \mathcal{A}'$. Restricted on the subalgebra $\mathcal{A}$, it is specified by

$$\kappa(u_{ij}) = u^*_{ji}.$$

The constructed κ represents *the antipode* morphism. It is important to observe that our quantum SU(2) is *not* interpretable as a subgroup of the quantum U(2). It is just a *projected* space.

Let us now present a different construction, based on the idea of 'fluctuating co-unit'. In such a formulation, we can construct a full group structure on a quantum SU(2). We start very similarly as in the previous construction, however we shall assume here that μ is simply a unitary complex number (not just a central unitary element).

As we have already observed, the unitarity condition for the matrix

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \tag{4.38}$$

is equivalent to the following commutation relations:

$$\alpha\gamma = \bar{\mu}\gamma\alpha \qquad \alpha^*\gamma^* = \bar{\mu}\gamma^*\alpha^* \qquad \alpha\gamma^* = \mu\gamma^*\alpha \qquad \alpha^*\gamma = \mu\gamma\alpha^* \tag{4.39}$$

$$\alpha^*\alpha + \gamma^*\gamma = 1 \tag{4.40}$$

$$\alpha\alpha^* = \alpha^*\alpha \qquad \gamma\gamma^* = \gamma^*\gamma. \tag{4.41}$$

At the higher-order levels, we shall postulate the following natural collectivity relations:

$$\alpha_i\gamma_j = \bar{\mu}\gamma_j\alpha_i \quad \alpha_i^*\gamma_j^* = \bar{\mu}\gamma_j^*\alpha_i^* \quad \alpha_i\gamma_j^* = \mu\gamma_j^*\alpha_i \quad \alpha_i^*\gamma_j = \mu\gamma_j\alpha_i^* \tag{4.42}$$

$$\alpha_i\alpha_j^* = \alpha_j^*\alpha_i \qquad \alpha_i\alpha_j = \alpha_j\alpha_i \qquad \gamma_i\gamma_j^* = \gamma_j^*\gamma_i \qquad \gamma_i\gamma_j = \gamma_j\gamma_i. \tag{4.43}$$

where $i \neq j$. Here are the explicit expressions for the coproduct

$$\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = u \rightsquigarrow u_1u_0^*u_2 = \begin{pmatrix} \alpha_1 & -\mu\gamma_1^* \\ \gamma_1 & \alpha_1^* \end{pmatrix} \begin{pmatrix} \alpha_0^* & \gamma_0^* \\ -\bar{\mu}\gamma_0 & \alpha_0 \end{pmatrix} \begin{pmatrix} \alpha_2 & -\mu\gamma_2^* \\ \gamma_2 & \alpha_2^* \end{pmatrix}$$

$$\alpha \rightsquigarrow \alpha_1\alpha_0^*\alpha_2 + \gamma_1^*\gamma_0\alpha_2 + \alpha_1\gamma_0^*\gamma_2 - \mu\gamma_1^*\alpha_0\gamma_2$$

$$\gamma \rightsquigarrow \gamma_1\alpha_0^*\alpha_2 - \bar{\mu}\alpha_1^*\gamma_0\alpha_2 + \alpha_1^*\alpha_0\gamma_2 + \gamma_1\gamma_0^*\gamma_2$$

$$\alpha^* \rightsquigarrow \alpha_1^*\alpha_0\alpha_2^* + \gamma_1\gamma_0^*\alpha_2^* - \mu\gamma_1\alpha_0^*\gamma_2^* + \alpha_1^*\gamma_0\gamma_2^*$$

$$\gamma^* \rightsquigarrow \gamma_1^*\alpha_0\alpha_2^* + \alpha_1\alpha_0^*\gamma_2^* + \gamma_1^*\gamma_0\gamma_2^* - \bar{\mu}\alpha_1\gamma_0^*\alpha_2^*$$

while the antipode is given by $u \rightsquigarrow u_0u^*u_0$ and the co-unit is given by simply shifting $u \rightsquigarrow u_0$.

As a matter of a straightforward verification, we can can prove that the standard quantum group associated to this 'fluctuating co-unit' version of quantum SU(2) is, in fact, the classical SU(2) group. This follows from the fact that

$$u_iu_j^* = \begin{pmatrix} a & c^* \\ c & a^* \end{pmatrix}$$

and so the unitarity of the above matrix implies that its entries mutually commute. They also commute between different matrices of the above form. So the entire realization is equivalent to the classical SU(2) group.

Remark 4.10. This example can be naturally embedded in a larger family of quantum SU(2) groups, where μ is an arbitrary non-zero complex number. However, in the case of non-unitary μ we have to deal with unbounded operators. Although the standard quantum group is still the classical SU(2).

4.5 Quantum Torus and Quantum Euclidean Plane

In this subsection we shall consider these two mutually related objects– quantum versions torus and Euclidean plane. They are both fundamentally connected with the historical and conceptual origins of quantum mechanics.

The quantum plane is, essentially, the Heisenberg algebra given by hermitian non-commutative coordinates p and q such that

$$pq - qp = \frac{\hbar}{i}. \tag{4.44}$$

The quantum torus is an exponential version of this plane, given by unitary elements U and V and a single relation

$$VU = zUV \tag{4.45}$$

where z is a (necessarily unitary) scalar.

We shall begin here by constructing a realization of quantum torus, in the form of a standard quantum (matrix) group. In this realization, the 'quantum deformation parameter' z can not simply be a number, but an active central element, exhibiting non-trivial behaviour under the action of diagrammatic morphisms.

Then we shall present an alternative formulation, based on a 'wandering co-unit'. Finally, a corresponding construction will be presented, for the quantum plane.

Let us consider a C*-algebra A generated by two unitaries U and V, and the following relations

$$UVU^{-1}V^{-1} = VU^{-1}V^{-1}U = V^{-1}UVU^{-1}. \tag{4.46}$$

This is equivalent to saying that the commutator $z = VUV^{-1}U^{-1}$ is a central element of A. As such, it reduces to a (unitary) scalar in every irreducible representation of A. In this sense, our quantum torus QT contains,

in a natural way, all standard realizations of rational and irrational rotation algebras, including the classical torus. In particular, the classical part of QT is naturally interpretable as the classical torus.

Let us define a collectivity structure for QT. In algebras A^{*n} let us define unitary objects

$$z_{ij} = V_i U_j V_i^* U_j^* \tag{4.47}$$

where $i, j \in \{1, \ldots, n\}$. Here we use simple lower indexes to denote the corresponding copies in the collectivity algebras. Clearly

$$z_{ii} = z_i \tag{4.48}$$

Let us consider the C*-ideal J_n generated by the following two sets of relations. At first,

$$z_{ij} U_k = U_k z_{ij} \qquad z_{ij} V_k = V_k z_{ij} \tag{4.49}$$

where $i, j, k \in \{1, \ldots, n\}$ are not all equal (for the relation to be a new relation). And also

$$U_i U_j = U_j U_i \qquad V_i V_j = V_j V_i. \tag{4.50}$$

It is clear that formulae

$$U \rightsquigarrow U_1 U_2 \qquad V \rightsquigarrow V_1 V_2 \tag{4.51}$$

consistently and uniquely define a unital *-homomorphism $\phi\colon A \to A_2 = A \star A / J_2$. Finally, if we define

$$u = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

then the space QT equipped with ϕ becomes a standard (matrix) quantum group.

Proposition 4.7. *The co-unit $\epsilon\colon A \to A$ and the antipode $\kappa\colon A \to A$ morphisms are given by*

$$\epsilon(U) = \epsilon(V) = 1 \qquad \kappa(U) = U^* \qquad \kappa(V) = V^* \tag{4.52}$$

and moreover we have

$$\phi(z) = z_{11} z_{12} z_{21} z_{22}. \tag{4.53}$$

Proof. The co-unit and antipode expressions are a trivial consequence of the general expression for matrix quantum groups. Expression (4.53) directly follow from the definition of z_{ij}, indeed

$$\phi(z) = \phi(VUV^*U^*) = V_1V_2U_1U_2V_2^*V_1^*U_2^*U_1^* = \\ = V_1U_1V_1^*U_1^*V_2U_2V_2^*U_2^*z_{21}z_{12} = z_{11}z_{22}z_{12}z_{21}.$$

It is worth observing that the unitaries z_{ij} generate *centers* of collectivity algebras A_n. □

It is also worth observing that the above mentioned centers behave like an autonomous structure, invariant under product and coproduct, and thus the entire diagrammatic category realization is possible. Specifically,

$$\phi_i(z_{ij}) = \begin{cases} z_{i,j+1}z_{i+1,j+1} & i < j \\ z_{ij}z_{i+1,j} & i > j \end{cases} \qquad \phi_j(z_{ij}) = \begin{cases} z_{i,j}z_{i,j+1} & i < j \\ z_{i+1,j}z_{i+1,j+1} & i > j \end{cases}$$

for $i \neq j$.

We can interpret these relations as the manifestation of an 'active' nature of the deformation parameter z. If we are to consider the whole group, it can not be a simple scalar. Quantum torus integrates in itself a variety of 'oscillating modes', corresponding to the scalar versions of the torus (where z assumes a concrete value). To have a complete group structure, we need at least some of them mixed together.

Let us now consider the 'standard' quantum torus, where z is a fixed unitary complex number. In this case it is necessary to use a fluctuating co-unit formulation, as explained in Subsection 4.3. We shall assume that Q generates an identical copy of the individual algebra, and use indexes 0 to refer to that copy C. So the diagrammatic morphisms will not act upon anything with the index 0.

Let us define the collectivity algebras A_n as generated by unitaries U_i and V_j where $i, j = 0, 1, \ldots, n$ and relations

$$U_iU_j = U_jU_i \qquad V_iV_j = V_jV_i \qquad V_iU_j = zU_jV_i. \tag{4.54}$$

A general formula (4.16) for the coproduct gives

$$\phi_i(U_i) = U_iU_0^*U_{i+1} \qquad \phi_i(V_i) = V_iV_0^*V_{i+1}. \tag{4.55}$$

Proposition 4.8. *It is easy to see that the constructed structure is a matrix quantum group based on* (u, Q). *The co-unit and the antipode are given by*

$$\epsilon(U) = U_0 \qquad \epsilon(V) = V_0 \qquad \kappa(U) = U_0 U^* U_0 \qquad \kappa(V) = V_0 V^* V_0. \quad (4.56)$$

Proof. At first, we have to verify the consistency of the above definition of the coproduct. We see that $U_1 U_0^* U_2$ and $V_1 V_0^* V_2$ satisfy the same defining relations as U and V. We can also replace U_i and V_i in the above defining collectivity relations, by $\phi_i(U_i)$ and $\phi_i(V_i)$, making shifts $i \mapsto i+1$ $j \mapsto j+1$ where appropriate, and the resulting relations are a simple consequence of the collectivity relations. The expressions for the co-unit and the antipode are a special case of the formulas (4.17). □

Remark 4.11. The higher-order algebras A_n have large centers. Indeed, all the elements of the form $U_i U_j^*$ and $V_i V_j^*$ where $i \neq j$ are central. For irrational parameters z, these elements actually generate the centers of A_n. If z is rational, the centers also include individual algebras central generators U_i^k and V_j^k where k is the grade of z.

Remark 4.12. The standard matrix group associated to the torus, is the classical torus group U(1) × U(1).

Let us now turn our attention to the quantum Euclidean plane. At the purely algebraic level. Let $\mathcal{A}$ be a *-algebra generated by hermitian elements p, q and a single relation (4.44), where $\hbar$ is a positive number. Let us define the higher-order *-algebras $\mathcal{A}_n$ as generated by hermitian elements p_k and q_l where

$$p_k q_l - q_l p_k = \frac{\hbar}{i} \quad (4.57)$$

and $k, l = 0, 1, \ldots, n$. We shall also assume that the p-generators mutually commute, as well as the q-generators.

The group structure is specified by the following coproduct assignments

$$p \rightsquigarrow p_1 - p_0 + p_2 \qquad q \rightsquigarrow q_1 - q_0 + q_2. \quad (4.58)$$

This consistently defines a unital *-homomorphism $\phi\colon \mathcal{A} \to \mathcal{A}_2$, which is compatible with the higher-order algebras.

Proposition 4.9. *In such a way we obtain a quantum group. The co-unit and the antipode are given by*

$$\epsilon(p) = p_0 \qquad \epsilon(q) = q_0 \qquad \kappa(p) = 2p_0 - p \qquad \kappa(q) = 2q_0 - q. \tag{4.59}$$

Proof. To prove the existence of the coproduct, observe that $p_1 - p_0 + p_2$ and $q_1 - q_0 + q_2$ satisfy the same defining relations as p and q. We can also replace p_k and q_l in the relations defining collectivity algebras, by $p_k - p_0 + p_{k+1}$ and $q_l - q_0 + q_{l+1}$, making index shifts $k \mapsto k+1$ or $l \mapsto l+1$ where appropriate. The resulting relations are a simple consequence of the collectivity relations. The expressions for the co-unit and the antipode follow from the general theory, and it can be also straightforwardly verified. □

Remark 4.13. The standard quantum group associated to the quantum plane, is isomorphic to the classical additive group $\mathbb{R} \oplus \mathbb{R}$.

The presented construction can easily be generalized to higher dimensional Weyl algebras. And an analogous construction can be developed for arbitrary Clifford algebras. Indeed, let us consider a finite dimensional vector space V equipped with a symmetric non-degenerate form $g\colon V \times V \to \mathbb{C}$. The Clifford algebra relations

$$xy + yx = 2g(x, y) \tag{4.60}$$

are naturally refinable into higher-order collectivity relations:

$$x_i y_j + y_j x_i = 2g(x, y), \tag{4.61}$$

where $x, y \in V$. The coproduct, co-unit and antipode are given by

$$\phi(t) = t_1 - t_0 + t_2 \qquad \epsilon(t) = t_0 \qquad \kappa(t) = 2t_0 - t, \tag{4.62}$$

where, as always we identify the individual algebra with its collectivity copy carrying index One.

Our definition of quantum plane, preserves the Euclidean symmetry: The group E(2) of Euclidean transformations (semidirect product of translations and rotations) naturally acts by automorphisms of the *-algebra defined by (4.44). If we define the quantum plane in the following way

$$zz^* = qz^*z \tag{4.63}$$

where $q \neq 1$ is a positive number then, clearly, the Euclidean symmetry is lost. The quantum space has only one point, corresponding to the 'origin' $z = 0$. The definition naturally extends to the collectivity structure, as

$$z_i z_j^* = q z_j^* z_i \qquad\qquad z_i z_j = z_j z_i. \tag{4.64}$$

And we can introduce the standard geometric quantum group structure by postulating $\phi\colon z \mapsto z_1 + z_2$. The unique point of the space plays the role of the co-unit, and the antipode is given by $\kappa\colon z \mapsto -z$.

Both examples can be naturally merged into

$$zz^* = qz^*z + \hbar, \tag{4.65}$$

assuming that $0 < q < 1$ and $\hbar > 0$. The classical points of this space form a circle of radius $\sqrt{\hbar/(1-q)}$. The collectivity structure is given by

$$z_i z_j^* = q z_j^* z_i + \hbar \qquad\qquad z_i z_j = z_j z_i. \tag{4.66}$$

The quantum group structure is of the 'fluctuating co-unit' type with the coproduct given by $\phi\colon z \mapsto z_1 - z_0 + z_2$.

4.6 Braided Commutative Hopf *-Algebras and Deformations

We can further generalize the last example of the quantum plane. Let us consider a standard braided Hopf *-algebra $\mathcal{A}$ such that the braid operator $\sigma\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is involutive, and the product map $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is σ-symmetric in the sense of $m\sigma = m$. We can then define the collectivity algebras as the corresponding braided tensor products of $\mathcal{A}$ with itself. In terms of our notation, the relations defining these algebras are of the form

$$[a]_i[b]_j = \sum_\alpha [b_\alpha]_j [a_\alpha]_i \qquad\qquad \sum_\alpha b_\alpha \otimes a_\alpha = \sigma(a \otimes b).$$

All *supergroups* are of this kind. We can furthermore consider a 'deformed' product $\tilde{m}$ on the vector space $\mathcal{A}$. Let us assume that such a product is fully compatible with the existing *-structure and the unit, and the braid operator σ (in the sense of fulfilling the pentagonal functoriality diagrams).

And finally, let us assume that the reinterpreted ϕ as $F\colon \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ is a *-homomorphism (relative to the algebra structure in the tensor products induced by σ).

Then, as it is easy to see, we can proceed as explained in the conclusion of Subsection 4.3 and build a 'fluctuating co-unit' quantum group structure on the quantum space given by $\mathcal{B}$.

4.7 Universal Envelopes of Lie Algebras

Let L be a complex Lie algebra, equipped with the product $[,]\colon L \times L \to L$. Let $\mathcal{A} = U_L$ be the associated universal enveloping algebra. By definition, it is the unital algebra built over L and relations

$$xy - yx = [x, y].$$

Let us define the collectivity algebras $\mathcal{A}_n$ by refining the above relations into

$$[x_i y_j - y_j x_i] + [x_j y_i - y_i x_j] = [x, y]_i + [x, y]_j, \tag{4.67}$$

where $i, j = 0, \ldots, n$ and, in order to simplify the notation, as in the previous examples we interpret the indexed symbols as the corresponding copy in the collectivity algebra.

Proposition 4.10. *The formula*

$$t \rightsquigarrow t_1 - t_0 + t_2$$

consistently defines a coproduct map $\phi\colon \mathcal{A} \to \mathcal{A}_2$, so that we have a quantum group structure. The co-unit and the antipode are given by

$$\epsilon(t) = t_0 \qquad \kappa(t) = 2t_0 - t.$$

Proof. We begin by verifying the consistency at the level of individual algebra $\mathcal{A}$. We compute

$$\begin{aligned} xy - yx \rightsquigarrow (x_1 - x_0 + x_2)(y_1 - y_0 + y_2) - (y_1 - y_0 + y_2)(x_1 - x_0 + x_2) \\ = [x, y]_1 + [x, y]_2 + [x, y]_0 + [x_1 y_2 - y_2 x_1] + \\ + [x_2 y_1 - y_1 x_2] - [x_1 y_0 - y_0 x_1] - [x_2 y_0 - y_0 x_2] \\ - [x_0 y_1 - y_1 x_0] - [x_0 y_2 - y_2 x_0] = [x, y]_1 + [x, y]_2 + [x, y]_0 + \\ + [x, y]_1 + [x, y]_2 - [x, y]_1 - [x, y]_0 - [x, y]_2 - [x, y]_0 = [x, y]_1 + [x, y]_2 - [x, y]_0 \end{aligned}$$

which proves the consistency with the initial relations. Let us check the consistency with the higher-order relations (4.67). Let us substitute x_i with $\phi(x_i)$. We have

$$\begin{aligned}
[x_i y_j - y_j x_i] + [x_j y_i - y_i x_j] &\rightsquigarrow (x_i - x_0 + x_{i+1})y_k - y_k(x_i - x_0 + x_{i+1}) + \\
+ x_k(y_i - y_0 + y_{i+1}) - (y_i - y_0 + y_{i+1})x_k &= [x_i y_k - y_k x_i] + [x_k y_i - y_i x_k] + \\
+ [x_{i+1} y_k - y_k x_{i+1}] + [x_k y_{i+1} - y_{i+1} x_k] - [x_0 y_k - y_k x_0] - [x_k y_0 - y_0 x_k] & \\
= [x,y]_i + [x,y]_k + [x,y]_{i+1} + [x,y]_k - [x,y]_0 - [x,y]_k & \\
= [x,y]_i + [x,y]_{i+1} - [x,y]_0 + [x,y]_k &
\end{aligned}$$

where $k = j+1, j$ if $i < j$ and $i > j$ respectively. This proves the consistency of the coproduct as acting at the slot i.

Let us observe that ϵ is, essentially, the diagonal map, relativized to the index 0. And κ is obtained by applying the coproduct, flipping indexes 0 and 1 and projecting down by the diagonal map for 0 and 2:

$$t \rightsquigarrow t_1 - t_0 + t_2 \rightsquigarrow t_0 - t_1 + t_2 \rightsquigarrow 2t_0 - t.$$

So the co-unit and the antipode are well-defined unital *-homomorphisms. It is a matter of straightforward calculation to check that they indeed do what they are supposed to do. □

Remark 4.14. The commutation relations (4.67) are equivalent to saying that differences $x_i - x_j$ and $y_i - y_j$ mutually commute. The standard quantum group associated to this structure is simply the additive group of L, with the restricted higher-order commutativity relations: besides the commutation of $\tilde{x}_i = x_i - x_0$ and $\tilde{y}_i = y_i - y_0$, which reflects the base algebra, we also have the commutation of differences $x_i - x_j = \tilde{x}_i - \tilde{x}_j$ and $y_i - y_j = \tilde{y}_i - \tilde{y}_j$. This clearly projects to a classical solution—the symmetric algebra over L.

4.8 Quantum Differential Calculi & Finite Classical Groups

In this subsection we shall demonstrate how every finite group can be naturally included in a quantum differential envelope, which as such is a quantum geometric group. Our construction is canonical, in the sense that

it is exclusively based on an internal group data, which in its simplest realization is given by fixing an invariant subgroup of the given group.

The primary maps (and hence all morphisms in the diagrammatic category) will be represented as (necessarily unique and *-preserving) differential extensions of the initial diagonal and coproduct morphisms.

So let us begin by considering a finite group G and a first order bicovariant differential calculus Γ based on a subset S of $G \setminus \{\epsilon\}$. This is explained in detail in [4] and [5], in the context of quantum principal bundles. For the sake of completeness, we shall highlight here the principal structural elements.

The elements of G in a natural way label a basis in $\mathcal{A} = C(G)$. To $g \in G$ we can associate the Kronecker delta function whose support is $\{g\}$. We shall denote this function by the same symbol g. So the commutative algebra of functions is simply given by

$$\sum_{g \in G} g = 1 \qquad q \cdot g = \begin{cases} q & \text{when } q = g, \\ 0 & \text{otherwise.} \end{cases}$$

The coproduct and antipode act on these basis functions by

$$\phi(g) = \sum_{h \in G} h \otimes (h^{-1}g), \qquad \kappa(g) = g^{-1} \tag{4.68}$$

as it follows from the general definition that ϕ and κ are the pull-backs to functions of the group multiplication $G \times G \to G$ $((g,h) \mapsto gh)$ and group inversion $G \to G$ $(g \mapsto g^{-1})$, respectively.

The adjoint action is given by:

$$\mathrm{ad}(g) = \sum_{h \in G} (hgh^{-1}) \otimes h. \tag{4.69}$$

Of course, the elements $g \in G$ also can be interpreted via the map of evaluation at g as characters on $\mathcal{A}$.

The ideals in $\mathcal{A}$ are all *-ideals and are in a natural correspondence with subsets of G, consisting precisely of the points where all the functions of the ideal vanish. The ideals $\mathcal{R}$ satisfying $\mathcal{R} \subseteq \ker(\epsilon)$ determine the

left-covariant first-order quantum differential calculi over G. Such ideals correspond to subsets of G containing the neutral element ϵ. Or to put it slightly differently, there is a natural correspondence between these ideals $\mathcal{R}$ and subsets S of $G \setminus \{\epsilon\}$.

There exists a natural isomorphism $\Gamma_{inv} \leftrightarrow \ker(\epsilon)/\mathcal{R}$, and hence the elements $\{\pi(g) = [g] \mid g \in S\}$ form a basis in Γ_{inv}, the space of left-invariants of Γ. We are not interested in the trivial zero-dimensional calculus so S will always be non-empty.

This space is always a right $\mathcal{A}$-module in a natural way. The right module structure $\circ$ is explicitly given by

$$[g] \circ h = \begin{cases} [g] & \text{if } g = h, \\ 0 & \text{if } g \neq h. \end{cases} \tag{4.70}$$

The initial right $\mathcal{A}$-module structure on $\Gamma \leftrightarrow \mathcal{A} \otimes \Gamma_{inv}$ is given by

$$[g]h = (hg^{-1})[g]. \tag{4.71}$$

It is worth remembering that

$$[\epsilon] = -\sum_{s \in S} [s]. \tag{4.72}$$

The property of *-covariance $\kappa(\mathcal{R})^* = \mathcal{R}$ is equivalent to $S^{-1} = S$. On the other hand, the bicovariance property $\mathrm{ad}(\mathcal{R}) \subseteq \mathcal{R} \otimes \mathcal{A}$ is equivalent to $gSg^{-1} = S$ for all $g \in G$. In other words, the set S is invariant under the adjoint action of G. This means that S can be written as a (disjoint) union of conjugation classes.

In particular, we see that irreducible *-covariant bicovariant calculi are in a natural correspondence with the subsets of G which are either a single conjugate class that coincides with its inverse, or a union of two distinct mutually inverse conjugation classes.*

We shall here assume a complementary situation to this irreducibility: the set S, together with the neutral element ϵ, is a normal subgroup $H =$

*The notion of irreducibility understood as the non-existence of another non-trivial *-covariant and bicovariant calculus onto which the given calculus can be projected.

$S \cup \{\epsilon\}$ of G. Let us describe the collectivity structures. At the level of 0-order forms, we shall consider all possible functions on the n-fold direct product $G \times \cdots \times G$. Thus

$$\mathcal{A}^n = \overbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}^{n}.$$

At the level of first-order calculus, the n-th collectivity structure will be equipped with the calculus based on $S^n = (H \times \cdots \times H) \setminus \{\epsilon\}$ and we used the same symbol ϵ for the neutral element of all the groups.

Remark 4.15. In particular, if G is simple, we see that there is only one solution at the one-particle level: $H = G$. So the collectivity differential structures are based on $G^n = G \times \cdots \times G \setminus \{\epsilon\}$ the full n-fold direct products of G with itself minus $\{e\}$. In other words, they are all *universal.*

And for the higher-order calculus, we shall postulate the corresponding universal differential envelopes [2]–Appendix B. It is easy to reduce all the calculations of the higher-order collectivity structures, to the individual calculus. Here is an important example of quantum germ reductions. We shall use the squared indexes to indicate the corresponding copies.

Lemma 4.11. *We have*

$$\pi(a \otimes b) = \pi(a)_{[1]} \circ b_{[2]} + \epsilon(a)\pi(b)_{[2]} \tag{4.73}$$

for each $a, b \in \mathcal{A}$.

Proof. This is a direct consequence of the Leibniz rule for the $\circ$-structure on an arbitrary first-order calculus. □

The following gives explicit formulae to compute the diagonal map and coproduct, at the level of quantum germs.

Proposition 4.12. *Under the action of the diagonal map*

$$\pi[(g,h)] \rightsquigarrow \begin{cases} \pi(g) = \pi(h) & \text{if } g = h \\ 0 & \text{if } g \neq h \end{cases} \tag{4.74}$$

for every $g, h \in S$.

The coproduct map acts on germs as

$$\pi(a) \rightsquigarrow \kappa(a^{(1)})_{[2]}\pi(a^{(2)})_{[1]}a^{(3)}_{[2]} + \pi(a)_{[2]} = \kappa(a^{(1)})_{[2]}a^{(3)}_{[2]}\pi(a^{(2)} \otimes a^{(4)}) \quad (4.75)$$

for every $a \in \mathcal{A}$. *Another form of this action is*

$$\pi(g) \rightsquigarrow \sum_{x,u} x_{[2]}\pi[(u, x^{-1}u^{-1}xg)] \quad (4.76)$$

where we act on the group elements $g \in S$ *representing the germs.*

Proof. Let us first prove the coproduct action formulae. By the use of the definition of germs we obtain

$$\begin{aligned}\pi(a) = \kappa(a^{(1)})da^{(2)} \rightsquigarrow \left[\kappa(a^{(2)}) \otimes \kappa(a^{(1)})\right]\left[da^{(3)} \otimes a^{(4)} + a^{(3)} \otimes da^{(4)}\right] = \\ = \kappa(a^{(1)})_{[2]}\kappa(a^{(2)})_{[1]}(da^{(3)})_{[1]}a^{(4)}_{[2]} + \kappa(a^{(1)})_{[2]}\kappa(a^{(2)})_{[1]}a^{(3)}_{[1]}(da^{(4)})_{[2]} = \\ \kappa(a^{(1)})_{[2]}\pi(a^{(2)})_{[1]}a^{(3)}_{[2]} + \kappa(a^{(1)})_{[2]}\epsilon(a^{(2)})(da^{(3)})_{[2]} = \\ = \kappa(a^{(1)})_{[2]}\pi(a^{(2)})_{[1]}a^{(3)}_{[2]} + \pi(a)_{[2]}.\end{aligned}$$

On the other hand, by (4.73) we have

$$\begin{aligned}\kappa(a^{(1)})_{[2]}a^{(3)}_{[2]}\pi(a^{(2)} \otimes a^{(4)}) = \kappa(a^{(1)})_{[2]}a^{(3)}_{[2]}(\pi(a^{(2)})_{[1]} \circ a^{(4)}_{[2]}) + \\ + \kappa(a^{(1)})_{[2]}a^{(3)}_{[2]}\epsilon(a^{(2)})\pi(a^{(4)})_{[2]} = \kappa(a^{(1)})_{[2]}\pi(a^{(2)})_{[1]}a^{(3)}_{[2]} + \pi(a)_{[2]}.\end{aligned}$$

This proves (4.75). By acting on a single element $g \in S$ and using the explicit formula for the coproduct

$$\begin{aligned}\pi(g) \rightsquigarrow \kappa(g^{(1)})_{[2]}g^{(3)}_{[2]}\pi(g^{(2)} \otimes g^{(4)}) = \sum \kappa(y)_{[2]}x_{[2]}\pi[(u, v)] = \\ = \sum x_{[2]}\pi[(u, v)] = \sum_{x,u} x_{[2]}\pi[(u, x^{-1}u^{-1}xg)]\end{aligned}$$

where the first and second sums are over (x, y, u, v) and (x, u, v) such that $yuxv = g$ and $x^{-1}uxv = g$ respectively.

Finally, to apply the diagonal map on germs $\pi[(g, h)]$ means equalizing the first and the second component in the double expanded sum. This reduces to the single germ $\pi(g)$ if the term-by-term equalization is possible, that is, if $g = h$. Otherwise, it goes to zero. □

Lemma 4.13. *The antipodal morphism acts as follows:*

$$\pi(g) \rightsquigarrow \sum_x x\pi(x^{-1}g^{-1}x) \tag{4.77}$$

for every $g \in G$.

Proof. We compute

$$\pi(g) = \kappa(g^{(1)})dg^{(2)} \rightsquigarrow g^{(1)}d\kappa(g^{(2)}) = g^{(1)}\kappa(g^{(3)})\pi\kappa(g^{(2)}) =$$
$$= \sum_{x,u,y}(x \cdot y^{-1})\pi(u^{-1}) = \sum_{x,u} x\pi(u^{-1}) = \sum_x x\pi(x^{-1}g^{-1}x)$$

where the first and second sums are over (x, u, y) and (x, u) such that $xuy = g$ and $xux^{-1} = g$ respectively. □

Remark 4.16. In particular, we see that the antipodal extensibility as such implies that S must be invariant under conjugations (the standard bicovariance condition).

Remark 4.17. Our class of examples can be extended, without any substantial changes, to include calculi based on subsets S which are unions (excluding ϵ) of normal subgroups of G.

Remark 4.18. The presented constructions can be further generalized to arbitrary geometric quantum groups (including, of course, the classical groups), where we can start from a given first-order differential calculus Γ and construct the whole higher-order calculus, which is itself a quantum geometric group, a kind of universal geometric differential envelope of the initial data given by the group G and its first-order differential calculus.

A The Simplest Non-Abelian Cyclic Group

The group $B(2, 3)$ is defined as the quotient of $\mathcal{F}_2$ over the normal subgroup consisting of products of third powers of elements from $\mathcal{F}_2$. Here is an explicit construction of the group. Let $\mathbb{T}$ be the Abelian subgroup of $\mathbb{Z}_3 \times$

$\mathbb{Z}_3 \times \mathbb{Z}_3$ consisting of triples with vanishing sum of 3 coordinates. It is the kernel of the sum homomorphism

$$0 \to \mathbb{T} \rightarrowtail \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \twoheadrightarrow \mathbb{Z}_3 \to 0$$

The permutation group S_3, and in particular $\mathbb{Z}_3$, acts by permutation automorphisms of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. And $\mathbb{T}$ as a whole is invariant under these automorphisms. We then have

$$B(2,3) \backsimeq \mathbb{T} \ltimes \mathbb{Z}_3,$$

the semidirect product of $\mathbb{Z}_3$ and $\mathbb{T}$. The isomorphism is established by assigning to the generators of $\mathcal{F}_2$ any two non-commuting elements (a, b) of the semidirect product. For example

$$a \leftrightarrow (1,-1,0) \qquad b \leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here is the list of all elements of $B(2,3)$ in terms of generators. We can also consider it as an abstract specification:

$$\mathbb{T} = \{\epsilon, a, a^{-1}, bab^{-1}, ba^{-1}b^{-1}, abab^{-1}, a^{-1}ba^{-1}b^{-1}, aba^{-1}b^{-1}, a^{-1}bab^{-1}\}$$
$$\mathbb{T}b = \{b, ab, a^{-1}b, ba, ba^{-1}, aba, a^{-1}ba^{-1}, aba^{-1}, a^{-1}ba\}$$
$$\mathbb{T}b^{-1} = \{b^{-1}, ab^{-1}, a^{-1}b^{-1}, bab, ba^{-1}b, abab, a^{-1}ba^{-1}b, aba^{-1}b, a^{-1}bab\}.$$

We can also say that $B(2,3)$ is the simplest non-commutative group in which all cube powers trivialize.

The commutant of the group is isomorphic to $\mathbb{Z}_3$, and in the above representation consists of all vectors of $\mathbb{T}$ with equal coordinates. The commutant is also the center of the group.

B Quantum Determinants & Unitary Triples

We shall here consider realizations of the commutation relations for our quantum U(2) group, in an arbitrary *-algebra $\mathcal{X}$. We shall call such realized

triples (μ, Δ, u) unitary triples in $\mathcal{X}$. So if (μ, Δ, u) is a unitary triple, then its inverse $(\bar{\mu}, \Delta^*, u^{-1})$ is a unitary triple, too. We can always consider a *trivial* unitary triple, given by

$$\mu = \Delta = 1 \qquad u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition B.1. We shall say that two unitary triples are compatible, and write $(\mu_1, \Delta_1, u_1) \sim (\mu_2, \Delta_2, u_2)$ iff the relations from the 2-collectivity algebra of quantum U(2), hold. It is clear that the introduced relation is symmetric.

Proposition B.1. *The relation $\sim$ is reflexive: if (μ, Δ, u) is a unitary triple in $\mathcal{X}$, then $u \sim u$.*

Proof. If we identify both triples with (μ, Δ, u) then the commutation relations project to the commutation relations for (μ, Δ, u). □

Proposition B.2. *If (μ_1, Δ_1, u_1) and (μ_2, Δ_2, u_2) are compatible then $u_1 \sim u_2^*$, too. In other words, all the 4 triples–the initial two and their inverses are all mutually compatible. In particular, every unitary triple (μ, Δ, u) is compatible with its inverse triple $(\bar{\mu}, \Delta^*, u^*)$.*

Proof. We shave to check that if we replace one triple by its inverse, the commutation relations (4.28)–(4.34) remain the same. A straightforward calculation shows that the compatibility relations are invariant. □

Proposition B.3. *If (μ_1, Δ_1, u_1) and (μ_2, Δ_2, u_2) are compatible unitary triples then (μ, Δ, u) with $\mu = \mu_1\mu_2$, $\Delta = \Delta_1\Delta_2$ and $u = u_1u_2$ is a unitary triple.*

Proof. We have

$$u_1u_2 = \begin{pmatrix} \alpha_1\alpha_2 - \mu_1\gamma_1^*\Delta_1\gamma_2 & -\mu_1\gamma_1^*\alpha_2^* - \mu_2\alpha_1\gamma_2^*\Delta_1^* \\ \gamma_1\alpha_2 + \alpha_1^*\Delta_1\gamma_2 & \alpha_1^*\alpha_2^* - \mu_2\gamma_1\gamma_2^*\Delta_1^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Delta_1\Delta_2 \end{pmatrix}$$

and thus to proceed with the proof we have to verify that the first matrix is of a unimodulary type with $\mu = \mu_1\mu_2$. A direct calculation gives for diagonal elements

$$(\alpha_1\alpha_2 - \mu_1\gamma_1^*\Delta_1\gamma_2)^* = \alpha_2^*\alpha_1^* - \bar{\mu}_1\gamma_2^*\Delta_1^*\gamma_1 = \alpha_1^*\alpha_2^* - \mu_2\gamma_1\gamma_2^*\Delta_1^*$$

as well as

$$\mu_1\mu_2(\gamma_1\alpha_2 + \alpha_1^*\Delta_1\gamma_2)^* = \mu_1\mu_2\alpha_2^*\gamma_1^* + \mu_1\mu_2\gamma_2^*\Delta_1^*\alpha_1 = \mu_1\gamma_1^*\alpha_2^* + \mu_2\alpha_1\gamma_2^*\Delta_1^*$$

for off-diagonal elements. It remans to verify the commutation relations with the determinant $\Delta_1\Delta_2$. It is easy to see that it commutes with the diagonal elements, because of the mutually compensating presence of two gammas. On the other hand,

$$\Delta_1\Delta_2(\gamma_1\alpha_2 + \alpha_1^*\Delta_1\gamma_2)\Delta_2^*\Delta_1^* = \bar{\mu}_2^2\Delta_1(\gamma_1\alpha_2 + \alpha_1^*\Delta_1\gamma_2)\Delta_1^* = \\ = (\bar{\mu}_1\bar{\mu}_2)^2(\gamma_1\alpha_2 + \alpha_1^*\Delta_1\gamma_2)$$

Hence the product determinant also obeys the correct relations and we conclude that the product is a unitary triple. □

Definition B.2. The unitary triple (μ, Δ, u) is called the product of triples (μ_1, Δ_1, u_1) and (μ_2, Δ_2, u_2).

Proposition B.4. *Let (μ_1, Δ_1, u_1), (μ_2, Δ_2, u_2) and (μ_3, Δ_3, u_3) be mutually compatible unitary triples. Then*

$$(\mu_1, \Delta_1, u_1)(\mu_2, \Delta_2, u_2) \sim (\mu_3, \Delta_3, u_3)$$

in other words every triple is compatible with (both) products of the other two.

Proof. We have to check the α-γ relations, the $\gamma - \gamma$ relations, the commutation of alphas, and the mutual actions of the determinants. This is a straightforward and easy calculation.

Thus, the product $(\mu_1, \Delta_1, u_1)(\mu_2, \Delta_2, u_2)$ is compatible with (μ_3, Δ_3, u_3) and due to the completely symmetric appearance of 3 triples, we conclude that each one of them is compatible with both products of the other two. □

If we now consider a collection $\mathcal{C}$ of mutually compatible unitary triples, we can naturally proceed by extending the system, by including the inverses if they are not already included in $\mathcal{C}$, and products. In such a way we obtain a sequence

$$\mathcal{C} = \mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n \subseteq \cdots \mathcal{C}_\infty = \cup_{k\in\mathbb{N}}\mathcal{C}_k$$

of mutually compatible triples, where $\mathcal{C}_{n+1}$ is obtained from $\mathcal{C}_n$ by adding all products from $\mathcal{C}_n$. The union $\mathcal{C}_\infty$ is a group, the product being given by the product of triples. The trivial triple is the neutral element. The inverse triple is the group inverse. By construction $\mathcal{C}_\infty$ is the minimal group containing $\mathcal{C}$.

C From Classical to Quantum: A Cross Product

We shall now outline a general construction of an interesting class of quantum groups, which are based on classical groups and certain automorphisms, and yet sufficiently 'quantum' to provide non-trivial examples.

Let $\mathcal{A}$ be a commutative Hopf *-algebra. We shall denote as usual, by ϵ, κ and ϕ the counit, antipode and the coproduct, and imagine $\mathcal{A}$ as consisting of polynomial functions over a classical Lie group G. We shall assume that a *-automorphism $\delta\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ is given such that several important symmetry conditions hold.

Our first assumption is that $\mathcal{A} = \mathcal{A}\otimes\mathbb{C}$ is invariant for both δ and its inverse, and that the restriction $\delta\colon \mathcal{A} \to \mathcal{A}$ is such that the diagram

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\delta} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{A} & \xrightarrow[\delta]{} & \mathcal{A} \end{array} \tag{C.1}$$

is commutative. Here $m\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ is the product map. Let us consider the n-fold tensor products $\mathcal{A}^{\otimes n}$ where $n \geq 2$. For every $k \in \{1,\ldots,n\}$ we can introduce a *-automorphism $\delta_k\colon \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n}$ via the following diagrams

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\iota_{kl}} & \mathcal{A}^{\otimes n} \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta_k} \\ \mathcal{A}\otimes\mathcal{A} & \xrightarrow[\iota_{kl}]{} & \mathcal{A}^{\otimes n} \end{array} \tag{C.2}$$

where $\iota_{kl}\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}^{\otimes n}$ are canonical embeddings, moving the tensor factors into k-th and l-th positions of $\mathcal{A}^{\otimes n}$ respectively.

Remark C.1. Clearly, the maps δ_k are uniquely determined by the above formulae, and δ itself can be understood as δ_1 on $\mathcal{A}^{\otimes 2}$. Our notation is a bit informal here, because it does not reflect the dependence on n. However, it is justified by the definition of the maps δ_k which basically repeat the action of the initial δ, all it matters is the position in $\mathcal{A}^{\otimes n}$ of the invariant $\mathcal{A}$, the first factor in $\mathcal{A}\otimes\mathcal{A}$. We shall use interchangeably δ and δ_1 on all $\mathcal{A}^{\otimes n}$. A natural common playground for all these automorphisms is given by

$$\mathcal{A}^{\infty} = \lim_{\to} \mathcal{A}^{\otimes n}$$

the inductive limit algebra. All δ_k act as *-automorphisms of $\mathcal{A}^{\infty}$, and form a discrete group $\mathcal{G}$. It is the union of increasing subgroups $\mathcal{G}_n$, corresponding to generators $\delta_1, \ldots, \delta_n$.

Our last two assumptions for δ express a compatibility with the group structure. The diagrams

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\delta} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle \mathrm{id}\otimes\phi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\ \mathcal{A}^{\otimes 3} & \xrightarrow[\delta]{} & \mathcal{A}^{\otimes 3} \end{array} \qquad \begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\delta} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle \phi\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \phi\otimes\mathrm{id}} \\ \mathcal{A}^{\otimes 3} & \xrightarrow[\delta\delta_2]{} & \mathcal{A}^{\otimes 3} \end{array} \tag{C.3}$$

are commutative.

Remark C.2. Note the subtle asymmetry between the left and right in the above diagrams.

We shall now extend the algebra $\mathcal{A}$ and its associated tensor powers $\mathcal{A}^{\otimes n}$ into non-commutative *-algebras, by using the standard cross product construction. Let Δ be a symbolic generator satisfying

$$\Delta b = \delta(b)\Delta \qquad \Delta^{-1} = \Delta^{*} \tag{C.4}$$

for each $b \in \mathcal{A} \otimes \mathcal{A}$. Let $\widetilde{\mathcal{A}}$ be the *-algebra obtained by extending $\mathcal{A}$ in this way. Similarly for each k, we can introduce a symbolic generator Δ_k such that

$$\Delta_k \psi = \delta_k(\psi)\Delta_k \qquad \Delta_k^{-1} = \Delta_k^* \tag{C.5}$$

for each $\psi \in \mathcal{A}^\infty$. We shall assume that generators Δ_k generate the same group, as their counterparts δ_k. Let $\widetilde{\mathcal{A}}^\infty$ be the *-algebra generated in this way. It is the inductive limit of the *-subalgebras

$$\widetilde{\mathcal{A}}^n = \mathrm{gen}\{\Delta_1, \ldots, \Delta_n, \mathcal{A}^{\otimes n}\} \tag{C.6}$$

The algebras $\widetilde{\mathcal{A}}^n$ are thus cross products between $\mathcal{G}_n$ and $\mathcal{A}^{\otimes n}$. The algebra $\widetilde{\mathcal{A}}^\infty$ is the cross product between $\mathcal{A}^\infty$ and the group $\mathcal{G}$ of its *-automorphisms.

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